Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Wind Turbine Blade Section Thickness
Part 2
Several Aspects of the Reference's Article 3.5[1] Discussed
De Moivre's Theorem and Complex Variables

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As mentioned in Part 1 of this series, detailed scrutiny and comment are being presented on material in this textbook reference relating to wind turbine blade section thickness. Some clarification is provided to the analysis in the book and, thereby, questions surface about the conclusions that are drawn therein. If blade section thickness by itself has an effect on the lift force then how may this effect be described.

Before looking at the derivations in Article 3.5 of the reference, it is helpful to understand some of the mathematics of which use is made within them. What is introduced at the outset is some elementary complex variable theory, which may be unfamiliar to many. Due to the reliance placed on it in the derivations especially for conformal mapping, it is important that it be understood.

Complex Variables

The curious quantity which is defined as the square root of -1 or √-1 has been found to have usefulness in solving physical problems, a surprise to everyone who becomes acquainted with it. In the ordinary real number system, no such quantity exists. So it has been given the name of "an imaginary number" and denoted by the letter *i*. Note that this number squared or *i*^2 is not imaginary and has the real value of -1 and this number to the fourth power or *i*^4 has the real value of 1, and so this "imaginary" number is not far from being "real".

Think of this quantity as being part of a nonexistent fourth dimension in space and a shortcut is taken through this never-never land in arriving at a correct solution to problems. Certainly only real numbers may be answers. The combination of a real variable added together with an imaginary variable is called a complex variable. The mathematics of complex variables has quite real usefulness despite the "imaginary" component of the variable.

De Moivre's Theorem

The quantity *z* may be written as:

$$z = (x + iy) = r(\cos\theta + isin\theta) = re^{i\theta}$$

in both Cartesian and polar coordinates and also as the exponential function with an imaginary angle exponent.

[1] Refers to Abbott, Ira H. and von Doenhoff, Albert E., *Theory of Wing Sections*, 1959, LOC 60-1601, Dover Publications, New York, Chapter 3, Article 3.5, pages 50-53.

De Moivre's Theorem states that:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

At first sight, this equality seems to be impossible since the exponential function is not periodic in nature as are the sine and cosine and certainly not limited by the value of 1 as the sine and cosine are but can readily go on to high values approaching infinity as its argument is increased. Despite this, the imaginary factor applied to the angle argument adjusts the values of the exponential to stay within bounds as is demonstrated in the proof given below. A proof of this theorem is ordinarily not seen in most textbooks and is quite easy to understand, so it was thought to be of interest to provide it here.

Proof

The sine, cosine, and exponential functions may be expressed each in terms of an infinite series consisting of powers of their arguments. Note that for the sine and cosine the arguments are assumed to be in radians. See standard engineering handbooks such as McNeese and Hoag[1]. These series are copied in below:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

Now substitute the angle θ in for the *x* in both the cosine and sine series and apply the imaginary factor *i* to the sine series:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$
$$i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \ldots$$

Meawhile the same may be done to the exponential function series:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$$

[1] McNeese, Donald C. and Hoag, Albert L., *Engineering and Technical Handbook*, 1957, LOC 57-6690, Prentice-Hall, Inc., Englewood Cliffs, New Jersey

In the second line above the powers of *i* have been calculated and applied to each term. It can be seen immediately that the first, third, fifth, and seventh terms of this exponential series are the same as the first four terms of the cosine series. The second, fourth, sixth, and eighth terms of this exponential series with the factor *i* are the same as the first four terms of the sine series. This process continues in the same manner as more terms are included. Therefore, adding together the cosine series and the sine series, as modified by the factor *i,* equals the exponential series and, since the series expressions are equal, then it follows that the function expressions are equal:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

Conformal Mapping Using Complex Variables

Mapping is the process of assigning to all the points in one plane locations in another plane that have different dimensional relationships with each other. A circle may be mapped as a square by the use of a suitable math formula and so on. When an airfoil shape is mapped to another airfoil shape, all the flow lines of the original airfoil shape come along as well and can (although not necessarily always) assume proper flowlines around the new airfoil shape. The mapping is normally accomplished by the use of complex variables with locations on the abscissa as the real variable and locations on the ordinate as the imaginary variable.

The Joukowski Conformal Transformation is the mapping algorithm used in the flow analysis herein. It is as follows:

$$\zeta = z + \frac{a^2}{z}$$

where *a* is the radius of a basic or reference circle with its center at the origin in the plane to be mapped and *z* is a complex variable function that describes another circle with a larger radius (or at least a different radius) than *a* and whose center may not be at the origin, which is the profile from which the the airfoil shape is mapped.

The mapping then converts this second circle, in comparing it to the reference circle, into an airfoil in the second, mapped plane. The airfoil has a rounded leading edge and a sharp trailing edge and has thickness and camber, all controlled by the values of the complex variable *z* and real valued parameter *a* entered.

The function *z* may be described with Cartesian coordinates or with polar coordinates:

$$\zeta = (x + iy) + \frac{a^2}{x + iy} = r(\cos\theta + i\sin\theta) + \frac{a^2}{r(\cos\theta + i\sin\theta)}$$

If polar coordinates are used, De Moivre's Theorem then may be applied to the mapping and so the above transformation would look as follows:

$$\zeta = re^{i\theta} + \frac{a^2}{re^{i\theta}}$$

The division of a real number by a complex number, as is required in the second term, is somewhat awkward to carry out. Although it most often appears this way, it may be converted to another, more readily calculable, form as follows:

$$\frac{a^2}{x+iy} = \frac{a^2(x-iy)}{(x+iy)(x-iy)} = \frac{a^2(x-iy)}{x^2 + i\cancel{x}y - i\cancel{x}y + y^2}$$
$$= \frac{a^2(x-iy)}{x^2+y^2} = \frac{a^2}{r^2}(x-iy)$$

Doing the same with the complex variable expressed as the exponential by means of De Moivre's Theorem, the expression is even simpler:

$$\frac{a^2}{re^{i\theta}} = \frac{a^2}{r}e^{-i\theta}$$

This also equals in polar coordinates:

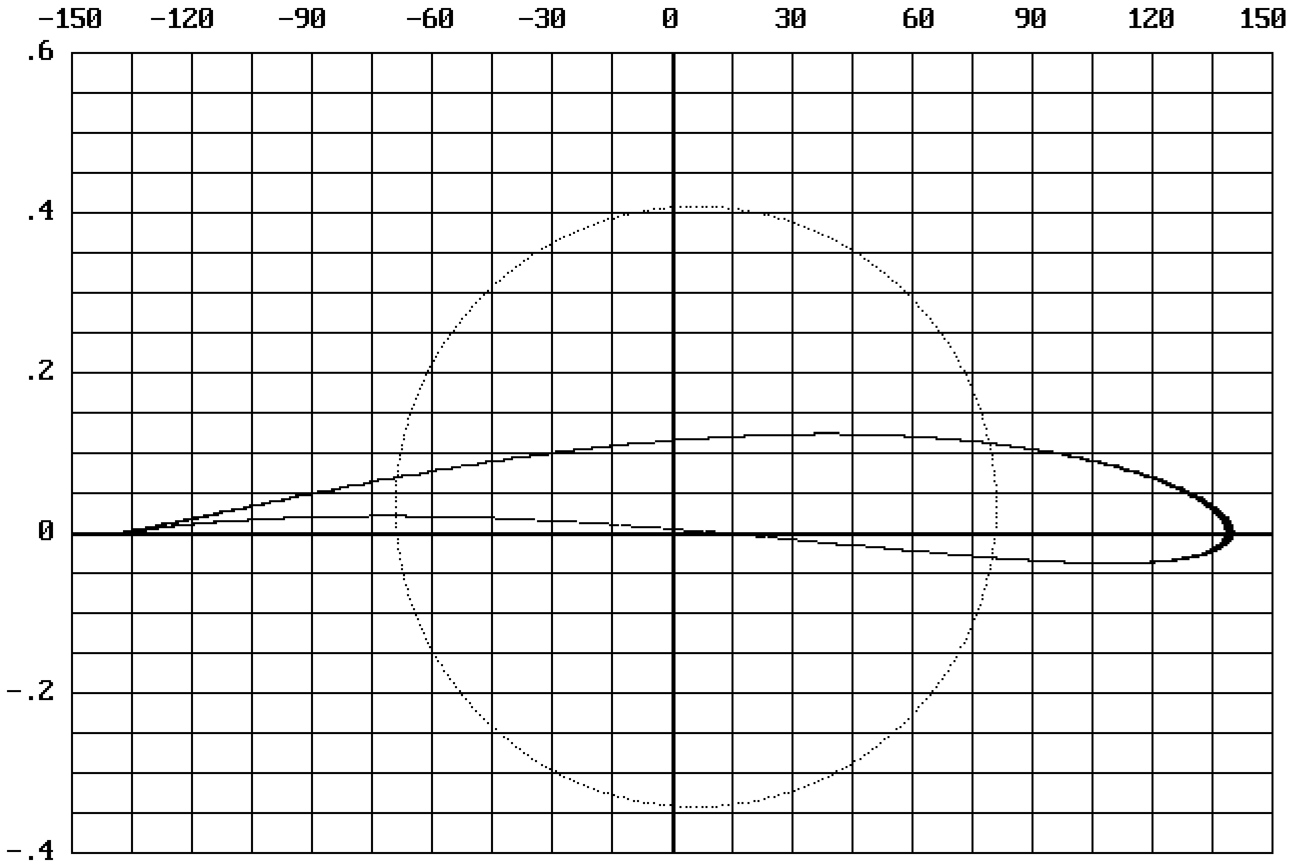
$$\frac{a^2}{r}e^{-i\theta} = \frac{a^2}{r}(\cos(-\theta) + i\sin(-\theta))$$
$$= \frac{a^2}{r}(\cos\theta - i\sin\theta)$$

Thus, the Joukowski Transformation may be expressed in alternative forms:

$$\zeta = (x+iy) + \frac{a^2}{(x^2+y^2)}(x-iy)$$
$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$$
$$= re^{i\theta} + \frac{a^2}{r}e^{-i\theta}$$

A Joukowski Conformal Transformation Mapping Using Complex Variables

Here below is an image taken from a Joukowski mapping that converts a circle to a wing or blade profile both of which may be seen. The camber as measured by the angle of the mean chord line above the x axis at the trailing edge is 10 degrees and the thickness-to-chord ratio is 10.4%. By varying the input complex variables parameters to the mapping algorithm other dimensions and ratios may be obtained:


-150
-120
-90
-60
-30
0
30
60
90
120
150
.6
.4
.2
0
-.2
-.4