Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Part 10

Derivation of the Lift Force Using ψ and ϕ with Verification of Conservation of Momentum

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As stated in Part 9, the superposition of a clockwise vortex flow field into the flow across a cylinder placed at right angles to a flow stream moving toward the cylinder from the left creates a circulation within the flow, adding to the flow across the top of the cylinder and inhibiting the flow across the bottom of the cylinder. An upward, vertical force is thereby created in the cylinder, the lift force.

A mathematical derivation of the lift force may now be presented. The derivation on page 45 of Abbott and von Doenhoff is copied in below:

The tangential component of velocity v' (positive counterclockwise) at the surface of the cylinder is obtained from Eq. (2.9) and the substitution of $r = a$.

$$v' = -2V \sin\theta + \frac{\Gamma}{2\pi a} \tag{2.30}$$

It is seen that the addition of the circulation Γ moves the points of zero velocity (stagnation points) from the positions $\theta = 0$ and π to the positions

$$\theta = \sin^{-1}\frac{\Gamma}{4\pi a V}$$

The pressure distribution about the cylinder may be found by applying Bernoulli's equation (2.5) along the streamline $\psi = 0$.

$$p + \frac{1}{2}\rho\left(4V^2\sin^2\theta - \frac{2V\Gamma\sin\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\right) = H \tag{2.31}$$

Setting

$$\frac{\Gamma}{2\pi a V} = K \tag{2.32}$$

the pressure coefficient S [Eq. (2.24)] becomes

$$S = 4\sin^2\theta - 4K\sin\theta + K^2 \tag{2.33}$$

The symmetry of Eq. (2.33) about the line $\theta = \pi/2$ shows that there can be no drag force. The lift on the cylinder can be obtained by integration, over the surface, of the components of pressure normal to the stream.

$$l = \tfrac{1}{2}\rho V^2 \int_0^{2\pi} Sa \sin\theta \, d\theta$$

$$= \tfrac{1}{2}\rho V^2 \int_0^{2\pi} (4a \sin^3\theta - 4aK \sin^2\theta + aK^2 \sin\theta)\, d\theta$$

$$l = \tfrac{1}{2}\rho V^2 ak[2\theta - \sin 2\theta]_0^{2\pi}$$

$$l = 2\rho V^2 ak\pi = \rho V\Gamma$$

It can be shown that the relation

$$l = \rho V \Gamma \tag{2.34}$$

is valid regardless of the shape of the body.[61]

This derivation in greater detail is as follows:

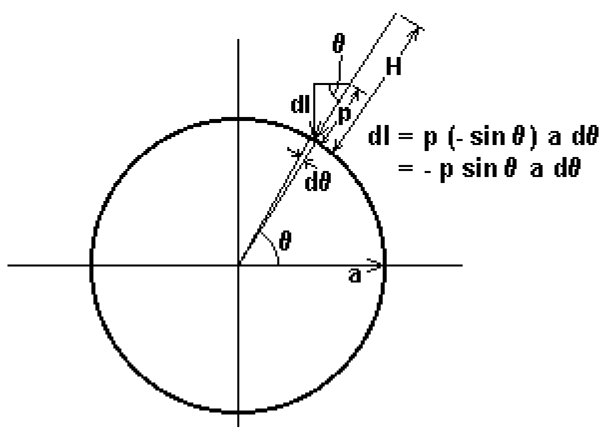
Lift Force Derivation

Start with a rewrite of equation 2.3.1 in the text above with Γ assumed to be positive in the clockwise direction. This is the Bernoulli Equation applied along the surface of the cylinder at the streamline ψ = 0, where the velocity is *v'* and the value of *r* is equal to *a*, the radius of the cylinder. The pressure is then isolated:

$$p(a,\theta) + \frac{1}{2}\rho\left(4V^2\sin^2\theta + \frac{2V\Gamma\sin\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\right) = H$$

where H is a constant equal to the flow field pressure at locations where velocities are equal to zero

$$p(a,\theta) = H - \frac{1}{2}\rho\left(4V^2\sin^2\theta + \frac{2V\Gamma\sin\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\right)$$



The increment of force, *dl*, at all points around the periphery of the cylinder in the vertical direction from the applied pressure then is given above. The total force upward positive in the vertical direction is found by substituting in the value for the pressure, *p*, and integrating around the cylinder surface from the angle θ equal to 0 to θ equal to 2π :

$$
\begin{aligned}
dl &= -p\sin\theta \;\, a \;\, d\theta \\
l &= -\int_0^{2\pi}\left(H - \frac{1}{2}\rho\left(4V^2\sin^2\theta + \frac{2V\Gamma\sin\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\right)\right)\sin\theta \;\, a \;\, d\theta \\
&= \left(\frac{\rho\Gamma^2}{8\pi^2 a} - Ha\right)\int_0^{2\pi}\cancelto{0}{\sin\theta}\; d\theta + \frac{\rho V\Gamma}{\pi}\int_0^{2\pi}\sin^2\theta \;\, d\theta + 2\rho V^2 a\int_0^{2\pi}\cancelto{0}{\sin^3\theta}\; d\theta \\
&= \frac{\rho V\Gamma}{\pi}\int_0^{2\pi}\sin^2\theta \;\, d\theta
\end{aligned}
$$

The integrals of the *sine* and *sine^3* are equal to zero by symmetry about the *x* axis. The integral of the *sine^2* can be found in integral tables, for example, McNeese and Hoag #95:

$$\int \sin^2 ax \;\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad where\ a = 1\ and\ x = \theta$$

The integration may now be completed:

$$
\begin{aligned}
l &= \frac{\rho V\Gamma}{\pi}\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right]_0^{2\pi} \\
&= \frac{\rho V\Gamma}{\pi}(\pi - 0 - 0 + 0) = \rho V\Gamma
\end{aligned}
$$

Conservation of Momentum

At this point in the analysis, only one half of the job has been done. A reaction to this lift force occurs in the flow field and it is not yet clear that it is equal and opposite to the force that has been so created. With a bit more effort to solve the calculus integrals necessary, it can be determined that such a reaction force is indeed created and it is equal to the same value in the negative direction, that is, $-\rho V\Gamma$.

First, it should be said that for this purpose it is better to use Cartesian coordinates.

Recall that:

$$\psi(x,y) = Vy\left(1 - \frac{a^2}{x^2+y^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{(x^2+y^2)^{1/2}}{a}$$

and

$$u(x,y) = \frac{\partial\psi}{\partial y} = V\left(1 - a^2\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)\right) + \frac{\Gamma}{2\pi}\frac{y}{(x^2+y^2)}$$

$$v(x,y) = -\frac{\partial\psi}{\partial x} = -\frac{2a^2Vxy}{(x^2+y^2)^2} - \frac{\Gamma}{2\pi}\frac{x}{(x^2+y^2)}$$

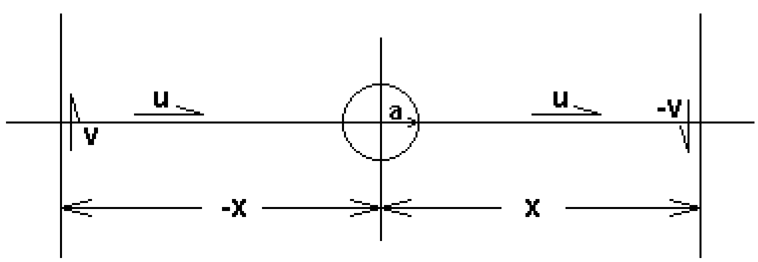
The flow field extends to infinity in all directions and so some thought must be given to what happens to the velocities in the limit, in particular, along the x axis:

$$As\ x \rightarrow -\infty\ or\ \infty$$

$$u(x,y) \rightarrow V\left(1 - \frac{a^2}{x^2}\right) + \frac{\Gamma}{2\pi}\frac{y}{x^2} \rightarrow V$$

$$v(x,y) \rightarrow -\frac{2a^2Vy}{x^3} - \frac{\Gamma}{2\pi}\frac{1}{x} \rightarrow 0$$

It is clear that the vertical velocities, *v*, at and near the x axis (*y* = 0) are always oriented upwards or positive to the left of the *y* axis and are always oriented downwards or negative to the right of the *y* axis as the flow proceeds from left to right. This indicates that a change in flow momentum may have occurred due to the circulation within the flow field. A measurement of this change can be obtained by selecting a place along the x axis to the left at a distance x from the *y* axis and integrating for the total vertical momentum up and down to infinity and doing the same on the right to determine the change.



The total force necessary in steady flow to create this change of flow momentum downwards then may be taken from Newton's Law expressed as a change of momentum:

$$l = \frac{dm}{dt}\nabla V\ where\ \frac{dm}{dt} = \rho V\int_{-\infty}^{\infty} dy$$

The ∇V or the "change in the velocity from one location to another" taken along each streamline is here the difference of the value of *v* in going from - x to + x. It is assumed that the x values are greater than or equal to the radius of the cylinder, *a*, and, for the moment, are the same, that is, the distances to the right and left of the cylinder balance. It also is seen within the expression for *v* above that the values of *v* are antisymmetrical across the *y* axis, that is, *v (-x) = - v (x).* So:

$$l = \rho V \int_{-\infty}^{\infty} \nabla v \; dy = \rho V \int_{-\infty}^{\infty} (v(-x) - v(x)) \; dy = \rho V \int_{-\infty}^{\infty} 2v(x) \; dy$$

The expression above for the value of *v(x, y)* may now be substituted in and the integration performed:

$$\begin{aligned} l &= \rho V \int_{-\infty}^{\infty} 2\left(-\frac{2a^2Vxy}{(x^2+y^2)^2} - \frac{\Gamma}{2\pi}\frac{x}{(x^2+y^2)}\right) dy \\ &= -2\rho V x \int_{-\infty}^{\infty} \left(2a^2V\left(\frac{y}{(x^2+y^2)^2}\right) + \frac{\Gamma}{2\pi}\left(\frac{1}{x^2+y^2}\right)\right) dy \\ &= -2\rho V x \left(2a^2V \int_{-\infty}^{\infty} \frac{y\,dy}{(x^2+y^2)^2} + \frac{\Gamma}{2\pi}\int_{-\infty}^{\infty} \frac{dy}{x^2+y^2}\right) \end{aligned}$$

The two integrals may be found in the integral tables, for example, McNeese and Hoag #s 64 and 63:

$$\int (ax^2+b)^n x \; dx = \frac{1}{2a}\frac{(ax^2+b)^{n+1}}{n+1} \quad where\ n=-2,\ a=1,\ b=x^2,\ x=y$$

$$\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\frac{x\sqrt{ab}}{b}\right) \quad where\ a=1,\ b=x^2,\ x=y$$

The integration may then be completed as follows:

$$\begin{aligned} l &= -2\rho V x \left(2a^2V\left[\frac{1}{2}\frac{(y^2+x^2)^{-1}}{-1}\right]_{-\infty}^{\infty} + \frac{\Gamma}{2\pi}\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{-\infty}^{\infty}\right) \\ &= -2\rho V x \left(a^2V\left[\frac{-1}{x^2+y^2}\right]_{-\infty}^{\infty} + \frac{\Gamma}{2\pi x}\left[\tan^{-1}\frac{y}{x}\right]_{-\infty}^{\infty}\right) \\ &= -2\rho V x \left(a^2V(0-0) + \frac{\Gamma}{2\pi x}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right) \\ &= -\cancel{2}\rho V \cancel{x}\left(\frac{\Gamma}{\cancel{2}\cancel{\pi}\cancel{x}}\cancel{\pi}\right) = -\rho V \Gamma \end{aligned}$$

The result is the same if two different values of the distance x are taken to the right and to the left.

Discussion

In considering the flow field at some distance from the cylinder in the analysis for the lift force for the first time (if it may be so claimed), the picture is now complete. While the cylinder is being forced upwards, the flow considered out to infinity is being forced downwards, though to a less perceptible degree due to its diffusion over the distances. In this way, momentum is conserved. If the distance beneath is interrupted by the earth at ground level, then an impact will be made on it from this flow as it completes its circle.

As for the mathematics used in this analysis, it should be said that it meticulously adheres to the assumption especially of irrotationality. In practice, air has some viscosity and so deviations from this model are to be expected. It is a good start and accomplishes much as it stands. The possibility always exists of adding further to the superpositions with more flow components such as those already incorporated within it to adjust it to suit practice. As such it may be considered to be only a beginning.

It now is made easier to study the energy aspects of lift on an airfoil and on the flow around it. While averaging was something of a guess in earlier models, the flow vectors can now be given better values by means of integrations through the now well described flow field. This becomes then something of particular value for wind energy.