

Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Part 9

Vortex Flow Added to Uniform and Doublet Flow Resulting In Flow Around a Cylinder with Circulation

Anthony Chessick, IntegEner-W, 2009

Now with the availability of the stream and cross stream functions for vortex flow as developed in Part 7, it becomes possible to adjust the flow model derived in Part 6 for a cylinder placed at right angles across a flow field to include the further attribute of flow field circulation.

This causes an unbalance between the flows over the top and below the bottom of the cylinder, thereby creating a force at right angles to the direction of uniform flow coming from the left, that is, a lift force in the upward direction. It also causes a vertical component of the velocities in the flow field upstream of the cylinder to be positive and downstream of the cylinder to be negative in reaction. The derivation of this force together with its reaction force in the flow field under conservation of momentum will be provided in the next Part.

Although the cylinder model used is far from a physical description of an airfoil for a wing in aviation or a blade in wind energy, the principles of how this force is produced in the cylinder model would be the same as those for such airfoils, if efforts in doing so such as these were continued for such applications. At this point, then, the process of how to move on with the mathematical tools provided herein, such as superposition of stream and cross stream functions, to cover such airfoils in an introductory manner should be now well enough understood.

Review of Flow Around a Cylinder for this Purpose

The stream and cross stream functions for flow around a cylinder, derived in Part 6 are:

$$\psi = Vr \left(1 - \frac{a^2}{r^2}\right) \sin \theta = Vy \left(1 - \frac{a^2}{(x^2 + y^2)}\right)$$
$$\phi = Vr \left(1 + \frac{a^2}{r^2}\right) \cos \theta = Vx \left(1 + \frac{a^2}{(x^2 + y^2)}\right)$$

The velocities are:

Polar Coordinates

$$u' = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = V \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -V \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

Cartesian Coordinates

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = V \left(1 + \frac{a^2(y^2 - x^2)}{(x^2 + y^2)^2} \right)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{-2Va^2xy}{(x^2 + y^2)^2}$$

Review of Vortex Flow for this Purpose

The stream and cross stream functions for vortex flow, derived in Part 7 are:

$$\psi = \frac{\Gamma}{2\pi} \ln \frac{r}{a} = \frac{\Gamma}{2\pi} \ln \frac{(x^2 + y^2)^{1/2}}{a}$$

$$\phi = -\frac{\Gamma}{2\pi} \theta = -\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$$

The velocities are:

Polar Coordinates

$$u' = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = 0$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\Gamma}{2\pi r}$$

Cartesian Coordinates

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\Gamma y}{2\pi(x^2 + y^2)}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -\frac{\Gamma x}{2\pi(x^2 + y^2)}$$

The Superposition

The superposed stream and cross stream functions for flow around a cylinder with circulation are:

$$\begin{aligned}\psi &= Vr \left(1 - \frac{a^2}{r^2}\right) \sin \theta + \frac{\Gamma}{2\pi} \ln \frac{r}{a} \\ &= Vy \left(1 - \frac{a^2}{(x^2 + y^2)}\right) + \frac{\Gamma}{2\pi} \ln \frac{(x^2 + y^2)^{1/2}}{a}\end{aligned}$$

$$\begin{aligned}\phi &= Vr \left(1 + \frac{a^2}{r^2}\right) \cos \theta - \frac{\Gamma}{2\pi} \theta \\ &= Vx \left(1 + \frac{a^2}{(x^2 + y^2)}\right) - \frac{\Gamma}{2\pi} \arctan \frac{y}{x}\end{aligned}$$

The velocities of the above superposed functions are:

Polar Coordinates

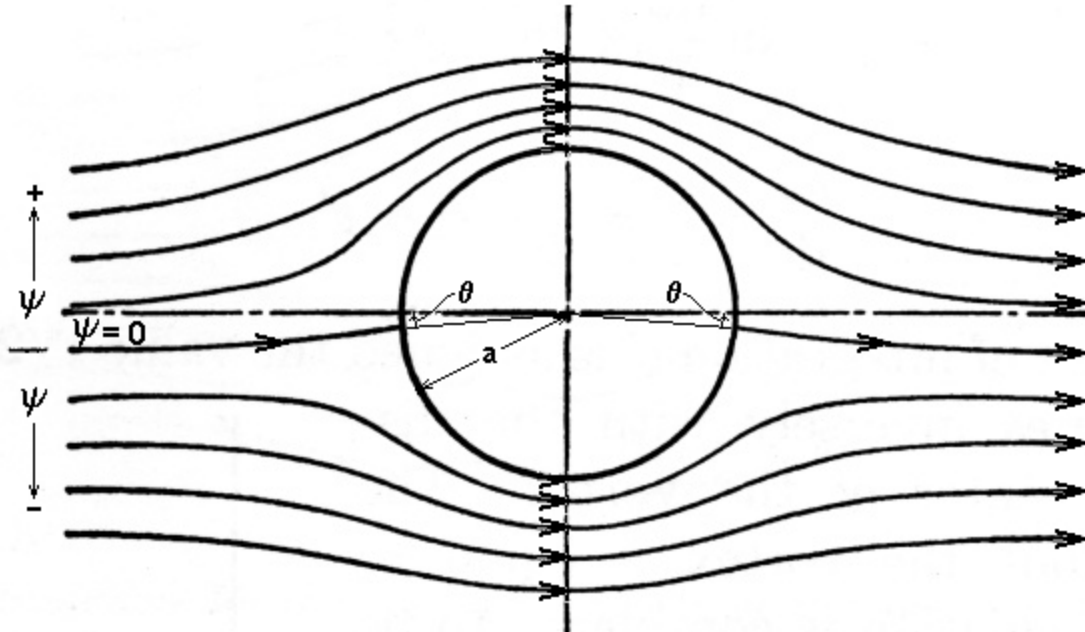
$$\begin{aligned}u' &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = V \left(1 - \frac{a^2}{r^2}\right) \cos \theta \\ v' &= -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -V \left(1 + \frac{a^2}{r^2}\right) \sin \theta - \frac{\Gamma}{2\pi r}\end{aligned}$$

Cartesian Coordinates

$$\begin{aligned}u &= \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = V \left(1 + \frac{a^2(y^2 - x^2)}{(x^2 + y^2)^2}\right) + \frac{\Gamma y}{2\pi(x^2 + y^2)} \\ v &= -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -\frac{2Va^2xy}{(x^2 + y^2)^2} - \frac{\Gamma x}{2\pi(x^2 + y^2)}\end{aligned}$$

These superposed functions satisfy the continuity and irrotationality conditions in both Cartesian and Polar coordinates.

The Flow Around a Cylinder with Circulation



The drawing above depicts the essential features of the flow. As earlier in Part 6 for Flow Around a Cylinder, only a limited number of stream function lines are included. Cross stream function lines are not included, nor are any of either of such lines within the cylinder circular boundary. Also not included are assumed dimensions and function values beyond zero of all flow lines.

The circulation in a clockwise direction introduced by the vortex component of the flowstream adds to the flow across the top of the cylinder and inhibits the flow beneath the cylinder. This drops the stagnation points on both sides, where the flow division takes place, to lower places on the cylinder boundary, signifying the relative flow mass differences. If the clockwise circulation were to be increased to a value such that its velocity at the cylinder boundary were to be equal to twice the value of V or greater, then the two stagnation points would drop farther and merge together at the very bottom point of the cylinder.

The Tangential Flow Velocities at the Cylinder Circular Boundary

In making use of the above flow velocities in polar coordinates, the value of the r dimension may be set equal to the radius of the cylinder, a , and the tangential flow at the cylinder circular boundary obtained:

$$v' = -V \left(1 + \frac{a^2}{a^2} \right) \sin \theta - \frac{\Gamma}{2\pi a} = -2V \sin \theta - \frac{\Gamma}{2\pi a}$$

The Continuity and Irrotationality Conditions

The continuity and irrotationality expressions may be derived for these superposed stream and cross stream functions in order to verify that the continuity and irrotationality conditions are met as is stated herein. An example of this process for just the irrotationality condition in Cartesian coordinates is provided below (the other three being readily derived as well):

To prove -

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} \left(V \left(1 + \frac{a^2(y^2 - x^2)}{(x^2 + y^2)^2} \right) + \frac{\Gamma y}{2\pi(x^2 + y^2)} \right) \\ &= Va^2 \left(\frac{(x^2 + y^2)^2 2y - (y^2 - x^2) 2(x^2 + y^2) 2y}{(x^2 + y^2)^4} \right) + \frac{\Gamma}{2\pi} \left(\frac{(x^2 + y^2) - y 2y}{(x^2 + y^2)^2} \right) \\ &= Va^2 y \left(\frac{(x^4 + 2x^2 y^2 + y^4) 2 + (x^4 - y^4) 4}{(x^2 + y^2)^4} \right) + \frac{\Gamma}{2\pi} \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \\ &= 2Va^2 y \left(\frac{3x^4 + 2x^2 y^2 - y^4}{(x^2 + y^2)^4} \right) + \frac{\Gamma}{2\pi} \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{2Va^2 xy}{(x^2 + y^2)^2} - \frac{\Gamma x}{2\pi(x^2 + y^2)} \right) \\ &= -2Va^2 y \left(\frac{(x^2 + y^2)^2 - x 2(x^2 + y^2) 2x}{(x^2 + y^2)^4} \right) - \frac{\Gamma}{2\pi} \left(\frac{(x^2 + y^2) - x 2x}{(x^2 + y^2)^2} \right) \\ &= -2Va^2 y \left(\frac{x^4 + 2x^2 y^2 + y^4 - 4x^4 - 4x^2 y^2}{(x^2 + y^2)^4} \right) - \frac{\Gamma}{2\pi} \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \\ &= 2Va^2 y \left(\frac{3x^4 + 2x^2 y^2 - y^4}{(x^2 + y^2)^4} \right) + \frac{\Gamma}{2\pi} \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \end{aligned}$$

hence -

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \equiv 0$$