Two Dimensional Wing and Blade Mathematical Theory
Detailing and Extending Material in Standard References

Part 8
Expanding The Determinations Of Pressure Distributions In Flow Fields Described Using ψ and ϕ

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Before going any further in completing a review of the analysis as presented in Abbott and von Doenhoff, it is appropriate to expand the determinations of pressure distributions provided on the pages therein to cover the entire flow fields rather than be limited just to a few flow boundaries, as is quite possible and a benefit obtainable form flow analyses making use of the stream and cross stream functions, ψ and ϕ.

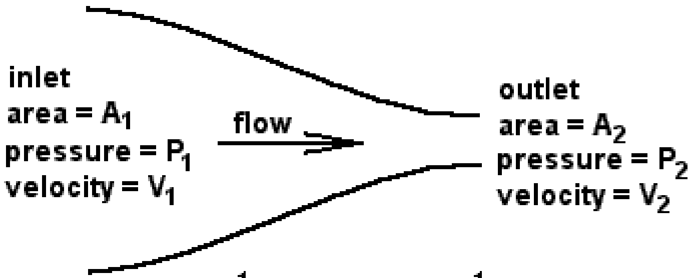
A Capsule Review

The use of the stream function, $\psi\ (x, y)$, and cross stream function, $\phi\ (x, y)$, in two dimensions, as derived from the Laplace equation under the assumptions of flow continuity and irrotationality, allows a mathematical description of flow fields that, by the technique of superposition, is a flexible method of modeling flows in many applications. Using them, lines may be drawn that represent actual flow stream lines and cross stream lines at right angles to the flow lines that, by their concentration, also indicate the velocity distributions of the flow. It should come as no surprise that these same functions may be used to determine the pressure distribution at all points all throughout the flow field as well.

Several constituent flow fields were defined and described earlier with the intention of superposing them, that is, adding together their stream and cross stream functions, into a description of the flow field desired, which is that of a circular cylinder cross section type of airfoil placed within a horizontal flow stream. At this point in the development of the analysis, here and as presented in the book, enough is available to go on to a derivation of the lift forces generated on the airfoil.

The Bernoulli Equation

As a first step in discussing the pressure field, it is important to understand the Bernoulli equation, written in below, and its use in determining pressures:



$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

Dimensionally, the Bernoulli equation is energy density, that is, energy per unit volume. The pressure term is, in fact, energy per unit volume and represents the work done in squeezing a bit of fluid mass into a small volume from a location of lower pressure outside of it. It is regained upon removing this mass back to a location outside of it, in terms of kinetic energy. It is characteristic of fluids to have a uniform energy density throughout them since mass within their volumes readily moves from high to low pressure areas, thus equalizing them. Elevation also is a factor to be considered but is neglected here.

In the drawing above, it can be shown that the inlet velocity, V1, has a maximum that can not be exceeded, given the inlet pressure, P1, and the ratio of the inlet to outlet flow area diameters, A1 to A2. This happens because the outlet pressure drops to zero at such an inlet velocity and so can not become any less than a perfect vacuum. The formula for determining this velocity is given below, derived from the Bernoulli equation and the equation of mass continuity for incompressible flow:

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho(V_2^2 - V_1^2)$$

Mass continuity at the inlet and outlet is:

$$V_1 A_1 = V_2 A_2$$

$$V_2 = \frac{V_1 A_1}{A_2}$$

Applying mass continuity to the above Bernoulli Equation:

$$P_1 - P_2 = \frac{1}{2}\rho\left(\left(\frac{V_1 A_1}{A_2}\right)^2 - V_1^2\right)$$

$$= \frac{1}{2}\rho V_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)$$

For an outlet pressure of zero, $P_2 = 0$

$$\left(\frac{P_1}{\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)}\right) = \frac{1}{2}\rho V_{1max}^2$$

Solving for V max at the inlet,

$$V_{1max} = \left(\frac{2P_1}{\rho\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)}\right)^{1/2}$$

Since the outlet pressure can not be any less than a perfect vacuum, the inlet velocity is limited to the maximum value given by this formula.

Uses For Pressure Distributions in Computer Fluid Flow Software

Much of this published material predates the advent of widespread computerization. The colorful images obtained from present day computational fluid dynamics software on computer screens depicting by their hues the pressures throughout flow spaces around airfoils, etc. even under time variations can be generated from the same basic mathematical treatment under review here.

The Derivation of Flow Field Pressure as a Function of ψ and ϕ

The derivation of flow field pressures starts with the Newton's equations for flow under pressure in two dimensions acting on flow particles under their material derivatives in steady motion. In Cartesian coordinates:

$$\frac{\partial p}{\partial x} = -\rho\frac{Du}{Dt} \qquad \frac{\partial p}{\partial y} = -\rho\frac{Dv}{Dt}$$

$$= -\rho\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right) \qquad = -\rho\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= -\rho\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v\right) \qquad = -\rho\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v\right)$$

$$\frac{\partial p}{\partial x}dx = -\rho\left(\frac{\partial u}{\partial x}u\ dx + \frac{\partial u}{\partial y}v\ dx\right) \qquad \frac{\partial p}{\partial y}dy = -\rho\left(\frac{\partial v}{\partial x}u\ dy + \frac{\partial v}{\partial y}v\ dy\right)$$

$$Since\ \frac{dx}{u} = \frac{dy}{v}\ and\ v\ dx = u\ dy,\ then$$

$$= -\rho\left(\frac{\partial u}{\partial x}u\ dx + \frac{\partial u}{\partial y}u\ dy\right) \qquad = -\rho\left(\frac{\partial v}{\partial x}v\ dx + \frac{\partial v}{\partial y}v\ dy\right)$$

$$= -\rho\ d\left(\frac{1}{2}u^2\right) \qquad = -\rho\ d\left(\frac{1}{2}v^2\right)$$

Adding both partial differentials for the total differential of the pressure,

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = -\frac{\rho}{2}d\left(u^2 + v^2\right) = -\frac{\rho}{2}d(V^2)$$

Integrating and setting the constant equal to H, which would be the pressure at locations where the velocity is zero,

$$p = H - \frac{\rho}{2}\left(u^2 + v^2\right)$$

The stream and cross stream functions may now be substituted in for the velocities,

$$= H - \frac{\rho}{2}\left(\left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2\right)$$

$$= H - \frac{\rho}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right)$$

In Polar coordinates:

$$\frac{\partial p}{\partial r} = -\rho\frac{Du'}{Dt} \qquad \frac{\partial p}{r\partial\theta} = -\rho\frac{Dv'}{Dt}$$

$$= -\rho\left(\frac{\partial u'}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u'}{r\partial\theta}\frac{r\partial\theta}{\partial t}\right) \qquad = -\rho\left(\frac{\partial v'}{r\partial\theta}\frac{r\partial\theta}{\partial t} + \frac{\partial v'}{\partial r}\frac{\partial r}{\partial t}\right)$$

$$= -\rho\left(\frac{\partial u'}{\partial r}u' + \frac{\partial u'}{r\partial\theta}v'\right) \qquad = -\rho\left(\frac{\partial v'}{r\partial\theta}v' + \frac{\partial v'}{\partial r}u'\right)$$

$$Since\ \frac{dr}{u'} = \frac{rd\theta}{v'}\ and\ v'dr = u'rd\theta,\ then$$

$$\frac{\partial p}{\partial r}dr = -\rho\left(\frac{\partial u'}{\partial r}u'dr + \frac{\partial u'}{r\partial\theta}u'rd\theta\right) \qquad \frac{\partial p}{r\partial\theta}rd\theta = -\rho\left(\frac{\partial v'}{r\partial\theta}v'rd\theta + \frac{\partial v'}{\partial r}v'dr\right)$$

$$= -\rho\ d\left(\frac{1}{2}u'^2\right) \qquad = -\rho\ d\left(\frac{1}{2}v'^2\right)$$

Similar to the case for Cartesian coordinates, now both partial differentials are added for the total differential of the pressure and integrated. The constant is set equal to H, which would be the pressure at locations where the velocity is zero.

$$p = H - \frac{\rho}{2}\left(u'^2 + v'^2\right)$$

The stream and cross stream functions may now be substituted in for the velocities,

$$= H - \frac{\rho}{2}\left(\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right)$$

$$= H - \frac{\rho}{2}\left(\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right)$$

The pressures may now be determined from the ψ or ϕ functions for any location throughout the flow fields and not just at flow boundaries, just as the flow vectors may be and also as is the determination of whether the flows as described satisfy the continuity and irrotationality conditions.

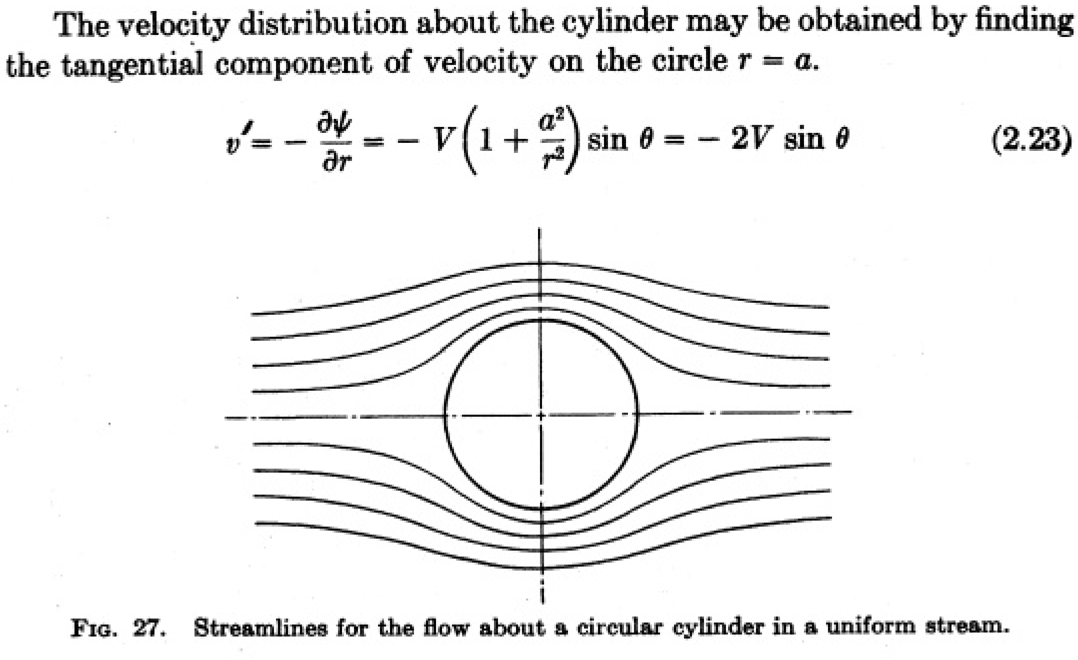
Conservation of Momentum

Now knowing the pressure distributions throughout the flow field, it becomes possible to gain a clearer picture of how the lift force is generated in the cylinder while conserving momentum, a question that is largely left unanswered in the theoretical treatment provided in this book and elsewhere in treatments of flows using the stream and cross stream analysis. One need only determine the pressures that are being created in the flow fields at lower levels to see their relative increases, thus demonstrating that a reaction to the lift forces is being created. Continued downward, these increased pressures eventually impact the ground, which then can be said to be the ultimate support for the lifting body as required under conservation of momentum.

Since the flow field descriptions within this stream and cross stream analysis extend out in all directions to infinity, a calculation of exactly what the reaction forces are to the lift force requires a detailed calculus integration that extends similarly to infinity, an analysis of some length that is to be covered in another Part to this series.

A Pressure and Flow Velocity Difficulty

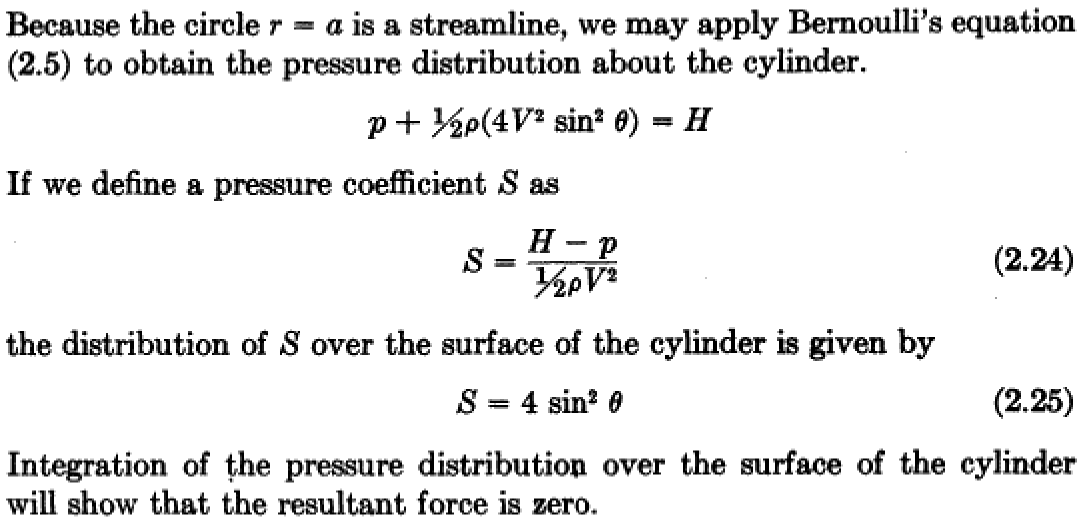
Copied in below is page 42 of the 1959 edition of Abbot and von Doenhoff. The discussion of flows and pressures about a circular cylinder in a uniform stream may be noted as having the difficulty that the velocity, V, is limited to a maximum given the application of an aviation airfoil, where H is assumed to equal atmospheric pressure and the velocity of the flow over the cylinder can reach values as great as 2V, readily dropping the surface pressure to zero. Certainly, the aircraft speed, V, is not so limited.



The velocity distribution about the cylinder may be obtained by finding the tangential component of velocity on the circle $r = a$.

$$v' = -\frac{\partial\psi}{\partial r} = -V\left(1+\frac{a^2}{r^2}\right)\sin\theta = -2V\sin\theta \qquad (2.23)$$

FIG. 27. Streamlines for the flow about a circular cylinder in a uniform stream.



Because the circle $r = a$ is a streamline, we may apply Bernoulli's equation (2.5) to obtain the pressure distribution about the cylinder.

$$p + \tfrac{1}{2}\rho(4V^2 \sin^2 \theta) = H$$

If we define a pressure coefficient S as

$$S = \frac{H - p}{\tfrac{1}{2}\rho V^2} \tag{2.24}$$

the distribution of S over the surface of the cylinder is given by

$$S = 4 \sin^2 \theta \tag{2.25}$$

Integration of the pressure distribution over the surface of the cylinder will show that the resultant force is zero.

This difficulty arises because, in practical applications, the cylinder cross section is moving with the velocity, V, and not the flow medium, which is standing still relative to the ground, the opposite of how the flows are being considered in this analysis. The flow over the cylinder at 2V would, then, in reality be air that is being propelled backwards from standing still to a velocity equal to V, that is, equal and opposite to the velocity of the cylinder that is in motion passing by. This would create quite a low pressure and can only be accomplished if the value of H is not atmospheric pressure but must be a pressure equivalent to the square of two times or more the value of V.

Another approach that may be taken is to assume that the local pressure, p, is a pressure that is to be considered to be relative to the pressures in the flow medium and may thereby fall to negative values. For purposes of further analysis, then, the value of H may be considered as having a relative value as well and becomes simply twice (1/2)ρV^2, that is, ρV^2. Then the pressures throughout the flow field vary between a minimun value of -ρV^2 and a maximum value of ρV^2. The values of the "pressure coefficient", S, (which, as defined, is a measure of pressure reduction) vary always from 4 at the locations of minimum pressure to 0 at the locations of maximum pressure and this is invariant for any values of H chosen, avoiding the difficulty otherwise present in the problem under the given assumptions and presentation.