

Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

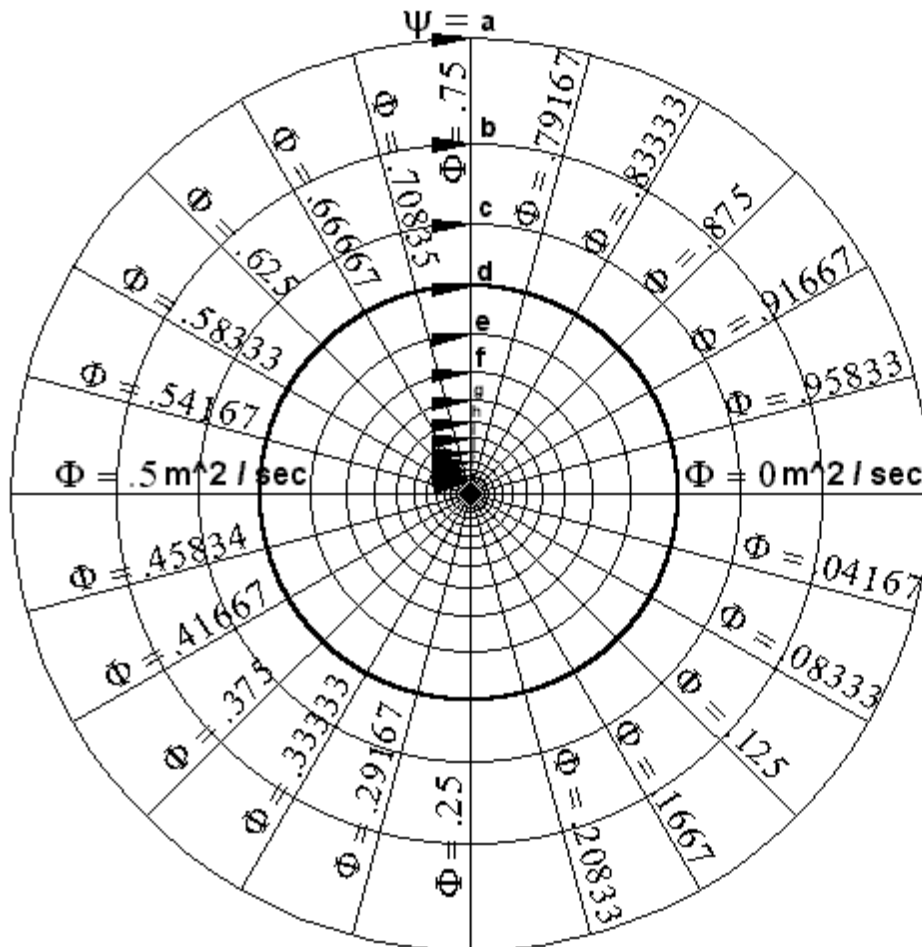
Part 7

Several Flow Fields Described Using ψ and ϕ : #4 - Vortex Flow

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Several flow field configurations are chosen to demonstrate the use of the stream (ψ) and cross stream (ϕ) functions in describing these fields. These were chosen as flow fields in particular to be superimposed upon each other in such a way as to form a composite flow field that resembles the flow field around an airfoil section of a blade or a wing. The fourth such flow field is covered in this part, the flow within a vortex flow field.

Flow Around a Vortex



ID	r (meters)	ψ (m ² /sec)	v' (m/sec)
a	2.1933	.1250	.0726
b	1.6881	.08333	.0943
c	1.2993	.04167	.1225
d	1.0000	0	.1592
e	.7697	-.04167	.2068
f	.5924	-.08333	.2687
g	.4560	-.1250	.3491
h	.3510	-.1667	.4536
i	.2700	-.2083	.5892
j	.2079	-.2500	.7655
k	.1600	-.2917	.9947
l	.1231	-.3333	1.293
m	.0948	-.3750	1.679
n	.0729	-.4167	2.183

The above is a drawing of the flow field. The flow begins on the x axis to the right of the origin and continues around in a clockwise direction until it overlaps itself back again on the x axis where it started. The description of the flow appears similar to that of the source and sink in Part 4 of this

series except that the ψ and ϕ functions are switched with each other. The flow is tangential and the flow velocities, which do not change along each streamline, are inversely proportional to their stream line distances from the origin. Accordingly, the rotational rates around the origin are inversely proportional to the square of the distances from the origin. High values of both the tangential velocities and rotation rates may be found near the origin, which is familiar vortex behavior.

Values were chosen for the flow quantities and the dimensions of the field itself as recorded both on the flow field and in the adjoining table. The flow stream circle identified as "d" was made bold to set it as having a radius of one meter and a value for the stream function, ψ , of zero.

The concept of irrotationality can even be extended to flows moving in a circle, as here. For the condition of irrotationality to be satisfied, it must be assumed that the fluid is inviscid and that no tangential stresses are presented by flow elements as they move past one another. It is readily seen that the integral of the flow velocities around any closed path within the flow field that does not include the origin, multiplied by the differential lengths of the flow path, is zero. If proscribed flow volumes within the flow of the above vortex were to be redirected to be in a straight path they would have a uniform flow velocity cross section and contain no angular velocities.

The Stream and Cross Stream Functions

First, find the stream function in polar coordinates by starting with the Laplace equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{Laplace Equation in Cartesian Coordinates}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad \text{Laplace Equation in Polar Coordinates}$$

The radial velocity, u' , in a vortex is zero and so the third term in the equation may be dropped and the equation multiplied by r :

$$u' = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0$$

$$\therefore \frac{\partial \psi}{\partial \theta} = 0 \text{ and } \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

and so the Laplace Equation reduces to:

$$r \frac{d^2 \psi}{dr^2} + \frac{d\psi}{dr} = 0$$

This is the same as:

$$\frac{d}{dr} \left(r \frac{d\psi}{dr} \right) = 0$$

Then integrate twice and set the constants as follows:

$$r \frac{d\psi}{dr} = c_1$$

$$\frac{d\psi}{dr} = \frac{c_1}{r}$$

$$\psi = c_1 \ln r + c_2$$

$$\text{Let } c_1 = \frac{\Gamma}{2\pi} \text{ and } c_2 = c$$

$$\psi = \frac{\Gamma}{2\pi} \ln r + c$$

The velocities in polar coordinates are found by differentiating, noting also the equivalent cross stream function derivatives:

$$u' = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = 0$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\Gamma}{2\pi r}$$

From these the cross stream function in polar coordinates may be found by integration, whose constant may be set equal to zero:

$$\therefore \phi = -\frac{\Gamma}{2\pi} \theta$$

The stream and cross stream functions and their corresponding velocities along the x and y axes may now be found in Cartesian coordinates:

$$\text{Since } r = (x^2 + y^2)^{1/2} \text{ and } \theta = \arctan \frac{y}{x}$$

$$\psi = \frac{\Gamma}{2\pi} \ln(x^2 + y^2)^{1/2} + c$$

$$\phi = -\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\Gamma y}{2\pi(x^2 + y^2)}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -\frac{\Gamma x}{2\pi(x^2 + y^2)}$$

The Continuity and Irrotationality Conditions

In Polar Coordinates -

Irrotationality	Continuity
$\frac{\partial v'}{\partial r} + \frac{1}{r} \left(v' - \frac{\partial u'}{\partial \theta} \right) \stackrel{?}{=} 0$	$\frac{\partial u'}{\partial r} + \frac{1}{r} \left(u' + \frac{\partial v'}{\partial \theta} \right) \stackrel{?}{=} 0$
$-\frac{\Gamma}{2\pi r^2} + \frac{1}{r} \left(\frac{\Gamma}{2\pi r} - 0 \right) \stackrel{?}{=} 0$	$0 + \frac{1}{r} (0 + 0) \stackrel{?}{=} 0$
$-\frac{\Gamma}{2\pi r^2} + \frac{\Gamma}{2\pi r^2} \equiv 0$	$0 \equiv 0$

In Cartesian Coordinates -

Irrotationality
$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \stackrel{?}{=} 0$
$\frac{\Gamma}{2\pi(x^2 + y^2)} + \frac{-\Gamma y \ 2y}{2\pi(x^2 + y^2)^2} - \frac{-\Gamma}{2\pi(x^2 + y^2)} - \frac{\Gamma x \ 2x}{2\pi(x^2 + y^2)^2} \stackrel{?}{=} 0$
$2(x^2 + y^2) - 2(x^2 + y^2) \equiv 0$

Continuity
$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \stackrel{?}{=} 0$
$\frac{-\Gamma y \ 2x}{2\pi(x^2 + y^2)^2} + \frac{\Gamma x \ 2y}{2\pi(x^2 + y^2)^2} \equiv 0$