Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Part 6

Uniform and Doublet Flow Combined as Flow Around a Cylinder

Anthony Chessick, IntegEner-W, 2009

It now becomes possible to model an actual flow from common practice, a cylinder placed at right angles across a flow field in such a way that the stream lines in two dimensions must be diverted around it. It should be clear that even with this diversion in the center, the flow at some distance away from this obstacle will be largely unaffected, continuing to move from left to right in nearly straight lines, there being a growing impact on flow lines with approaches nearer the cylinder. The continuity and irrotationality theories support this description of the flow by means of superposing the stream and cross stream functions of the uniform flow and the doublet flow, each described earlier. This would then constitute an initial attempt, requiring further refinements to be added later, to make use of the stream and cross stream functions in combination to mathematically reflect flows over airfoils.

Review of Uniform and Doublet Flow for this Purpose

The stream and cross stream functions for uniform flow from left to right along the x-axis with no y-axis component are:

$$\psi = Vy = Vr\sin\theta \qquad \phi = Vx = Vr\cos\theta$$

The velocities of the uniform flow:

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = V$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$$

$$u' = \frac{\partial\psi}{r\partial\theta} = \frac{\partial\phi}{\partial r} = V\cos\theta$$

$$v' = -\frac{\partial\psi}{\partial r} = \frac{\partial\phi}{r\partial\theta} = -V\sin\theta$$

The stream and cross stream functions for doublet flow are:

$$\psi = \frac{-\mu y}{2\pi(x^2+y^2)} = \frac{-\mu\sin\theta}{2\pi r}$$

$$\phi = \frac{\mu x}{2\pi\left(x^2+y^2\right)} = \frac{\mu\cos\theta}{2\pi r}$$

The velocities of the doublet flow:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\mu}{2\pi}\left(\frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\right)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{-\mu x y}{\pi\left(x^2 + y^2\right)^2}$$

$$u' = \frac{\partial \psi}{r\partial \theta} = \frac{\partial \phi}{\partial r} = \frac{-\mu \cos\theta}{2\pi r^2}$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{r\partial \theta} = \frac{-\mu \sin\theta}{2\pi r^2}$$

The Superposition

The superposed stream and cross stream functions are:

$$\psi = \psi_{uniformflow} + \psi_{doubletflow} = Vy - \frac{\mu y}{2\pi(x^2 + y^2)}$$

$$= yV\left(1 - \frac{a^2}{(x^2 + y^2)}\right) = rV\sin\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$\phi = \phi_{uniformflow} + \phi_{doubletflow} = Vx + \frac{\mu x}{2\pi(x^2 + y^2)}$$

$$= xV\left(1 + \frac{a^2}{(x^2 + y^2)}\right) = rV\cos\theta\left(1 + \frac{a^2}{r^2}\right)$$

where:

$$a^2 = \frac{\mu}{2\pi V}$$

The velocities of the above superposed functions are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = V\left(1 + \frac{a^2(y^2 - x^2)}{(x^2 + y^2)^2}\right)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{-2Va^2 xy}{(x^2 + y^2)^2}$$

$$u' = \frac{\partial \psi}{r\partial \theta} = \frac{\partial \phi}{\partial r} = V\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{r\partial \theta} = -V\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

These superposed functions satisfy the continuity and irrotationality conditions in both Cartesian and polar coordinates.

Now it is seen in Part 5 that the flow from the doublet on the left is to the left, which is opposite to the direction of the flow of the uniform flow case, which is taken here as always to the right. However, the doublet flow to the left is moving with velocities that are being reduced with distance from the origin. At some points, the velocities of the doublet flow lines, so moving to the left and starting greater than that of the uniform flow, become equal in magnitude to the velocities of the uniform flow lines, so moving to the right, and, when combined in the superposition process, the two flows then cancel at these points, becoming zero. On the x axis, the distance to the left of the origin where the combined flow becomes exactly zero is the radius of a cylinder placed in the path of the flow. The flow moving from the left can then proceed to the right no longer and must move with a vertical component up or down in being enabled to continue.

In polar coordinates, the loci of points where the radial flow, *u',* within the combined flow fields is equal to zero becomes the periphery of a circle about the origin, i.e. the profile of the cylinder, with a radius equal to *a*:

If -

$$u' = V\cos\theta\left(1 - \frac{a^2}{r^2}\right) = 0$$

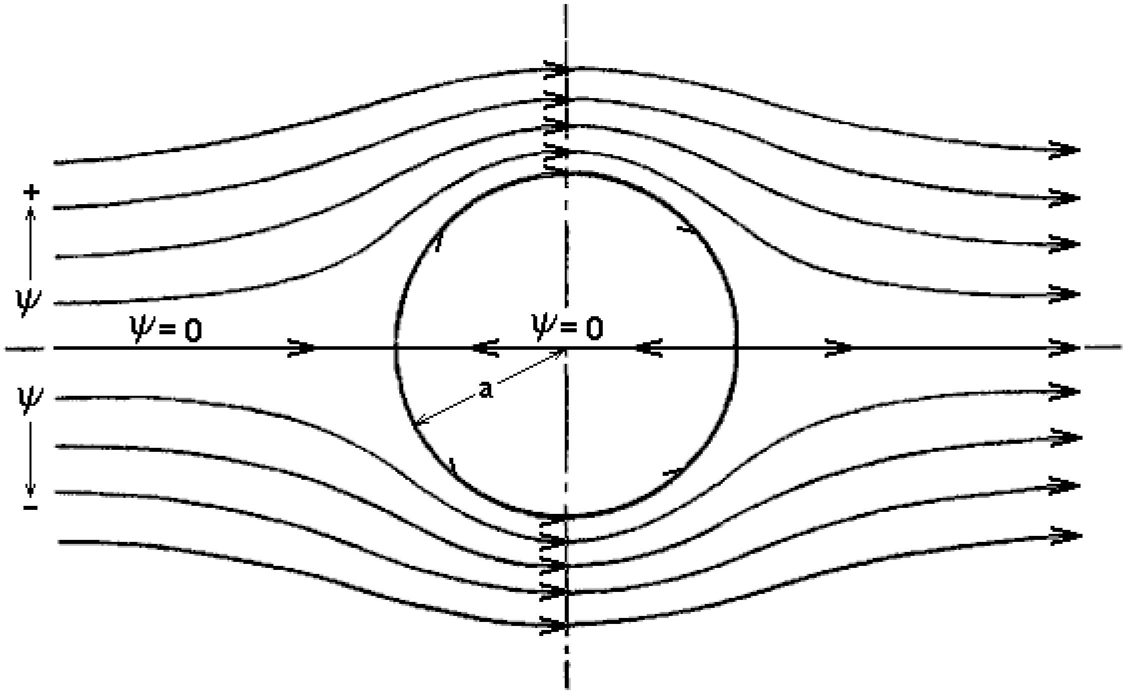
then -

$$1 - \frac{a^2}{r^2} = 0$$

$$r^2 = a^2$$

$$r = a$$

The Flow Around a Cylinder



The drawing above depicts the essential features of the flow. It is remarkable how with such little effort an accurate representation of this flow pattern is obtained by means of the superposition of the two stream and cross stream elements that are added together to form the relations necessary.

In simplifying the image - obtained above from combining the full and elaborate images of the uniform flow and the flow doublet - only a limited number of stream function lines are included. Cross stream function lines are not included. Also not included are assumed dimensions and function values beyond zero of all flow lines.

Stream flow lines do exist mathematically in the region within the circular cylinder profile. If drawn they would appear as a series of flattened circles within one another gradually becoming smaller, one such series in the top half of the cylinder with negative values and the other such series in the bottom half with positive values. All such lines in both halves converge to and at the origin point in the same way all stream lines converge to and at the origin point for the doublet. In the drawing above, flow arrows were added nominally for just the flow lines of $\psi = 0$ within the cylinder boundaries. The flow within this region thus recirculates, entirely separate from the flow outside of the circle, and is of no interest for purpose of the analysis being developed.

The Tangential Flow Velocities at the Cylinder Boundary

In making use of the above flow velocities in polar coordinates, the value of the *r* dimension may be set equal to the radius of the cylinder, *a*, and the tangential flow at the cylinder boundary obtained:

$$v' = -V\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

setting

$$r^2 = a^2$$

$$= -V\sin\theta(2) = -2V\sin\theta$$

This may then be used in an elementary way to find pressure distributions on the cylinder from the flow[1]. The Bernoulli Equation may be put to use in doing this task. Since the flow is symmetrical over the top and bottom of the cylinder, it is clear that forces arising from the pressure distributions will cancel each other out and no net force will be generated on the cylinder from the flow. This will soon change as will be seen in succeeding parts of this effort.

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[1] Although not recognized in ordinary treatments of this analysis, another type of pressure exists here, characterized by being non-isotropic and due entirely to localized accelerations within the moving fluid. The forces generated from these pressures will generally also net out and even within each side of the cylinder. A description of them is available in Batchelor, G.K., *An Introduction to Fluid Dynamics*, LOC #67-21953, Cambridge University Press, London and New York, 1967, Chapter 3, Article 3.3, Paragraph, "Mechanical definition of pressure in a moving fluid", pages 141 - 147.

The Continuity and Irrotationality Conditions

The continuity and irrotationality expressions may be derived for these superposed stream and cross stream functions in order to verify that the continuity and irrotationality conditions are met as is stated herein. An example of this process for just the continuity condition in Cartesian coordinates is provided below (the other three being readily derived as well):

To prove -

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

given -

$$u = V\left(1 + \frac{a^2(y^2 - x^2)}{(x^2+y^2)^2}\right)$$

$$v = \frac{-2Va^2xy}{(x^2+y^2)^2}$$

and so -

$$\begin{aligned}
\frac{\partial u}{\partial x} &= Va^2\left[\frac{(x^2+y^2)^2(-1)(2x) - (y^2-x^2)2(x^2+y^2)(2x)}{(x^2+y^2)^4}\right] \\
&= \frac{Va^2}{(x^2+y^2)^4}\left[-2x(x^4+2x^2y^2+y^4) - 4x(y^2x^2+y^4-x^4-x^2y^2)\right] \\
&= \frac{-Va^22x}{(x^2+y^2)^4}\left[x^4+\cancel{2x^2y^2}+y^4+2y^2x^2+2y^4-2x^4-\cancel{2x^2y^2}\right] \\
&= \frac{-2a^2xV}{(x^2+y^2)^4}(-x^4+2x^2y^2+3y^4)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= -2Va^2x\left[\frac{(x^2+y^2)^2 - y(2)(x^2+y^2)2y}{(x^2+y^2)^4}\right] \\
&= \frac{-2a^2xV}{(x^2+y^2)^4}\left[x^4+2x^2y^2+y^4-4x^2y^2-4y^4\right] \\
&= \frac{-2a^2xV}{(x^2+y^2)^4}\left(x^4-2x^2y^2-3y^4\right)
\end{aligned}$$

hence -

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2a^2xV}{(x^2+y^2)^4}(-\cancel{x^4}+2\cancel{x^2}y^2+3\cancel{y^4}+\cancel{x^4}-2\cancel{x^2}y^2-3\cancel{y^4}) \equiv 0 \quad \textbf{Q.E.D.}$$