Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Part 5

Several Flow Fields Described Using ψ and ϕ :

#3 - The Flow Doublet

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Several flow field configurations are chosen to demonstrate the use of the stream (ψ) and cross stream (ϕ) functions in describing these fields. These were chosen as flow fields in particular to be superimposed upon each other in such a way as to form a composite flow field that resembles the flow field around an airfoil section of a blade or a wing. The third such flow field is covered in this part, the flow at and near a flow doublet.

The Bicircular Flow of a Doublet

While stream and cross stream function flows can be said to likely represent physically possible flows closely if they meet the two conditions of continuity and irrotationality, this does not mean that such flows are not being asked to perform feats such as approaching infinite velocities and turning sharp corners on a dime. With this in mind, the source and sink flows of Part 4, with their high velocities at their center points, may be set side by side and in such a way as to form the flow doublet. The source may be placed on the left, producing flows moving to the left from it and the sink on its right, absorbing all the flows moving to it from the right. It may be imagined that in this arrangement, these are not a source and sink at all but just a location at which the flow streams converge and reach something like an infinite velocity within an infinitesimally small point. Even so, the doublet is another building block with which may be assembled by superposition composite flows of the desired geometries.

In Abbott and von Doenhoff[1], steps are taken mathematically to place the source and sink flow together as adjacent to one another and combined. In more formally taking limits as the distance between the two is reduced and the "strength" - *m* - of each is increased inversely:

$$\psi = \lim_{\substack{s \to 0 \\ m \to \infty}} \left(\frac{m}{2\pi} \arctan\left(\frac{-2ys}{x^2 + y^2 - s^2} \right) \right) = \frac{-\mu}{2\pi} \left(\frac{y}{x^2 + y^2} \right)$$

where $\mu = 2ms$ and an angle in radians that approaches zero is, of course, equal to its tangent in the limit.

[1] Abbott, Ira H. and von Doenhoff, Albert E., *Theory of Wing Sections*, 1959, Dover Publications, Inc., New York, LOC # 60-1601, Chapter 2, Article "2.6 Simple Two Dimensional Flows", Paragraph "c. Doublets", on page 41. Note also that the steps taken in making use of a trigonometric identity for the inverse tangents to obtain the first expression for the stream function above, not immediately obvious, may be verified by hand.

Then:

$$\psi = \frac{-\mu y}{2\pi\left(x^2+y^2\right)} = \frac{-\mu \cancel{r}\sin\theta}{2\pi r^{\cancel{2}}} = \frac{-\mu\sin\theta}{2\pi r}$$

and this is the flow doublet exactly (while an approximation of a flow doublet is shown in the drawing on the page). The stream function, ψ, so obtained may be checked and found to observe the continuity and irrotationality conditions in both Cartesian and polar coordinates, thus satisfying all necessary requirements placed on it.

The cross stream function, ϕ, as derived from the above stream function, is:

$$\phi = \frac{\mu x}{2\pi\left(x^2+y^2\right)} = \frac{\mu\cos\theta}{2\pi r}$$

A drawing of the doublet, as expressed by the above relations, using an assumed value for μ (and the drawing dimensions) is herewith provided below:

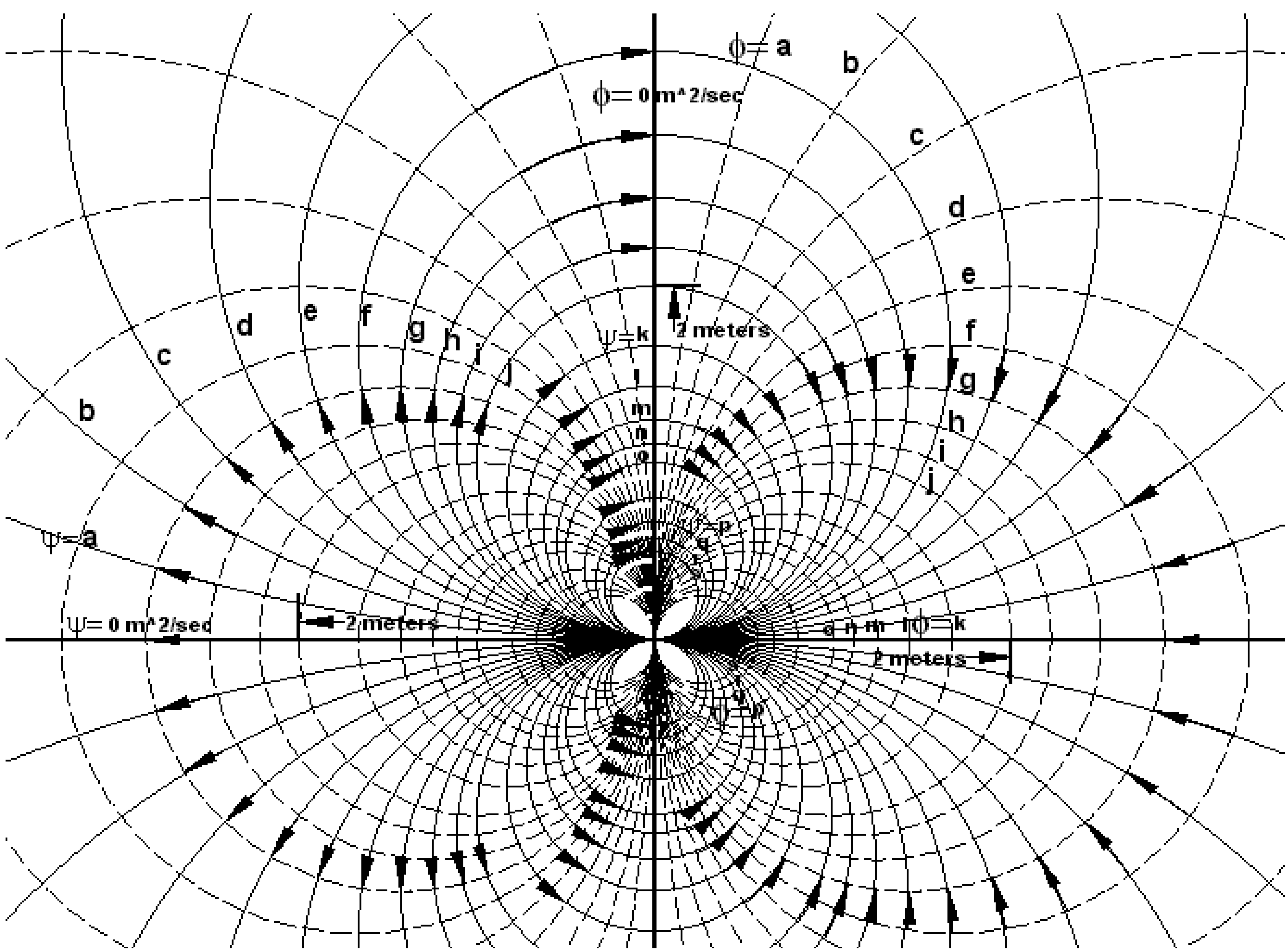

Table of Stream and Cross Stream Values for the Above Flow Doublet

For $\mu = 2\pi$ meters^3 / sec -

ID	Diam meters	ψ m^2/sec	ϕ m^2/sec
a	20	-.05	.05
b	10	-.1	.1
c	6.667	-.15	.15
d	5.0	-.2	.2
e	4.0	-.25	.25
f	3.333	-.3	.3
g	2.858	-.35	.35
h	2.50	-.4	.4
i	2.222	-.45	.45
j	2.0	-.5	.5

ID	Diam meters	ψ m^2/sec	ϕ m^2/sec
k	1.667	-.6	.6
l	1.429	-.7	.7
m	1.25	-.8	.8
n	1.111	-.9	.9
o	1.0	-1.0	1.0
p	.8	-1.25	1.25
q	.6667	-1.5	1.5
r	.5714	-1.75	1.75
s	.5	-2.0	2.0

The values of ψ below the x axis are mirror images of those above but are positive instead of negative.

The values of ϕ to the left of the y axis are mirror images of those to the right but are negative instead of positive.

The Velocities

The velocities in both Cartesian and polar coordinates are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\mu}{2\pi}\left(\frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\right)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{-\mu x y}{\pi\left(x^2 + y^2\right)^2}$$

$$u' = \frac{\partial \psi}{r \partial \theta} = \frac{\partial \phi}{\partial r} = \frac{-\mu \cos\theta}{2\pi r^2}$$

$$v' = -\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{r \partial \theta} = \frac{-\mu \sin\theta}{2\pi r^2}$$

The Continuity and Irrotationality Conditions

The four continuity and irrotationality conditions in both Cartesian and polar coordinates are met by these stream and cross stream functions but should be checked. Here just the irrotationality condition in polar coordinates is done as an example of this procedure:

$$\frac{\partial v'}{\partial r} + \frac{1}{r}\left(v' - \frac{\partial u'}{\partial \theta}\right) = 0$$

$$\frac{-\mu \sin\theta(-\cancel{2})}{\cancel{2}\pi r^3} + \frac{1}{r}\left(\frac{-\mu\sin\theta}{2\pi r^2} - \left(\frac{-\mu(-\sin\theta)}{2\pi r^2}\right)\right) \stackrel{?}{=} 0$$

$$\frac{\mu\sin\theta}{\pi r^3} - \frac{\mu}{2\pi r^3}(\sin\theta + \sin\theta) \stackrel{?}{=} 0$$

$$\frac{\mu}{\pi r^3}(\sin\theta - \sin\theta) \equiv 0 \quad \textbf{Q.E.D.}$$