Two Dimensional Wing and Blade Mathematical Theory

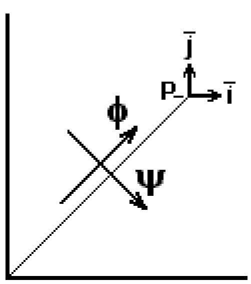
Detailing and Extending Material in Standard References

Part 3
Several Flow Fields Described Using ψ and ϕ :
#1 - The Uniform Flow Stream

Anthony Chessick, IntegEner-W, 2009

Several flow field configurations are now chosen to demonstrate the use of the stream (ψ) and cross stream (ϕ) functions in describing these fields. These flow fields were chosen as those in particular to be superimposed upon each other in such a way as to form a composite flow field that resembles the flow field around an airfoil section of a blade or a wing. The first is the simple field of uniform flow, covered here in this installment part. From henceforth the function "ϕ" will be identified as the "cross stream function" rather than the "velocity potential function" in keeping with comments made earlier.

Summary Definitions and Relations of ψ and ϕ in Parts 1 and 2[1]



$$d\psi = (u dy - v dx) = |\bar{V} \times \bar{ds}| = V ds \sin\theta$$

$$d\phi = (u dx + v dy) = \bar{V} \cdot \bar{ds} = V ds \cos\theta$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$\frac{\partial \psi}{\partial x} = -v \quad \frac{\partial \psi}{\partial y} = u \qquad \frac{\partial \phi}{\partial x} = u \quad \frac{\partial \phi}{\partial y} = v$$

The Gradients of Psi and Phi - $\nabla\psi$ and $\nabla\phi$

$$\nabla\psi = \left(\frac{\partial \psi}{\partial x}\bar{i} + \frac{\partial \psi}{\partial y}\bar{j}\right) \qquad \nabla\phi = \left(\frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j}\right)$$

$$= -v\bar{i} + u\bar{j} \qquad = u\bar{i} + v\bar{j}$$

Note that the absolute values of the gradients of both ψ and ϕ are equal to the absolute value of the flow velocity at any point. While the direction of $\nabla\phi$ is the same as the direction of flow, the direction of $\nabla\psi$ is 90 degrees counterclockwise to the direction of flow. This is the same for all rectangular coordinate systems.

---

[1] Math notations and examples of flow fields taken are generally the same as seen in Abbott, Ira H. and von Doenhoff, Albert E., *Theory of Wing Sections*, 1959, LOC #60-1601, Dover Publications, Inc., New York, Chapter 2

Laplace's Equation -

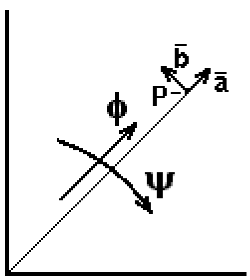
Derived from the irrotationality and continuity conditions as follows:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0 = \nabla^2 \psi \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \nabla^2 \phi$$

In writing the Laplacian for the stream function it is implicit in the definition of ψ that no fluid is being created or destroyed and the equation itself states that the flow is irrotational. In writing the Laplacian for the cross stream function it is implicit in the definition of ϕ that the motion is irrotational and the equation itself states that no fluid is being created or destroyed.

The Above Definitions and Relations of ψ and ϕ in Polar Co-ordinates -



$$d\psi = (u' r d\theta - v' dr) = |\bar{V} \times \bar{ds}| = V ds \sin\theta$$

$$d\phi = (u' dr + v' r d\theta) = \bar{V} \cdot \bar{ds} = V ds \cos\theta$$

$$d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta \qquad d\phi = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$$

$$\frac{\partial \psi}{\partial r} = -v' \quad \frac{\partial \psi}{\partial r} = r u' \qquad \frac{\partial \phi}{\partial r} = u' \quad \frac{\partial \phi}{\partial \theta} = r v'$$

The Gradients of Psi and Phi - $\nabla\psi$ and $\nabla\phi$ - in Polar Coordinates

$$\nabla \psi = \left(\frac{\partial \psi}{\partial r} \bar{a} + \frac{\partial \psi}{r \partial \theta} \bar{b} \right) = -v' \bar{a} + u' \bar{b} \qquad \nabla \phi = \left(\frac{\partial \phi}{\partial r} \bar{a} + \frac{\partial \phi}{r \partial \theta} \bar{b} \right) = u' \bar{a} + v' \bar{b}$$

Laplace's Equation in Polar Coordinates -

Derived from the irrotationality (left below) and continuity (right below) conditions in polar coordinates as follows:

$$\frac{\partial v'}{\partial r} + \frac{1}{r}\left(v' - \frac{\partial u'}{\partial \theta} \right) = 0 \qquad \frac{\partial u'}{\partial r} + \frac{1}{r}\left(u' + \frac{\partial v'}{\partial \theta} \right) = 0$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 = \nabla^2 \psi \qquad \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 = \nabla^2 \phi$$

In writing the Laplacian for the stream function in polar coordinates it is implicit in the definition of ψ that no fluid is being created or destroyed and the equation itself states that the flow is irrotational. In writing the Laplacian for the cross stream function in polar coordinates it is implicit in the definition of ϕ that the motion is irrotational and the equation itself states that no fluid is being created or destroyed.

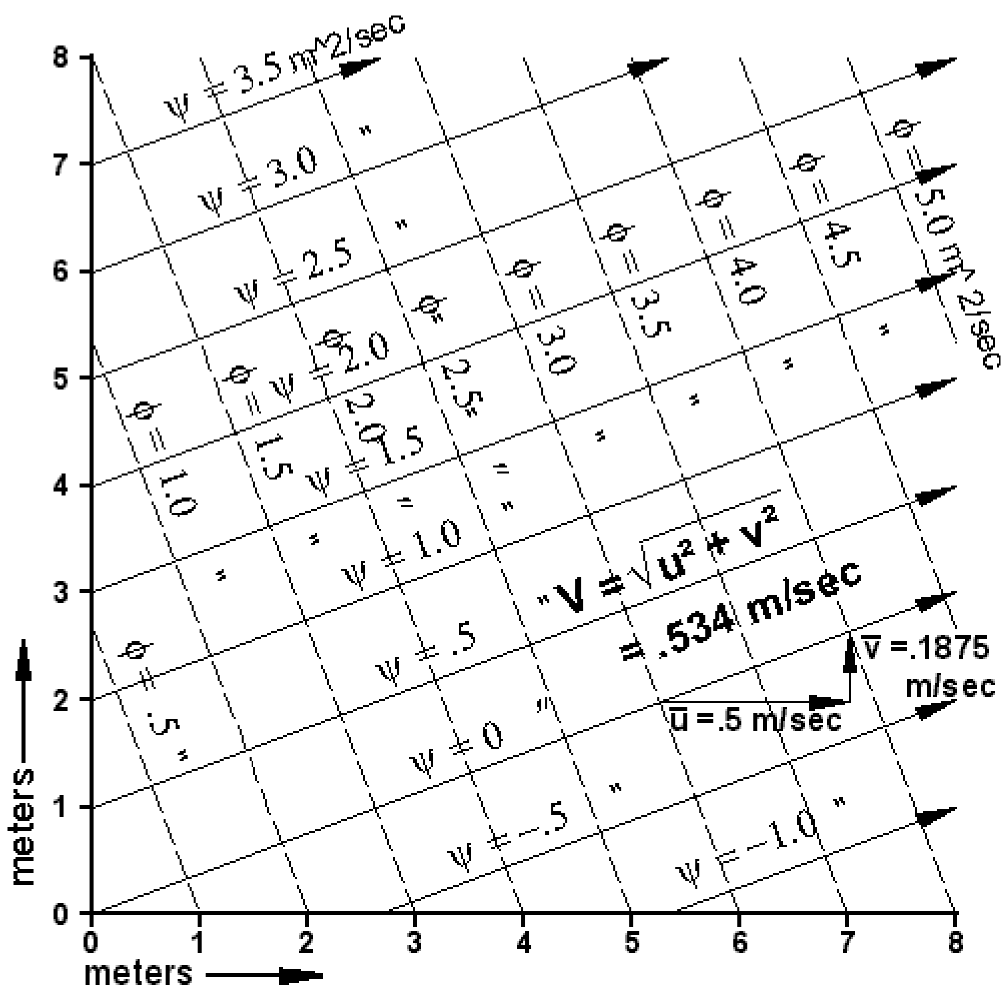
Uniform Flow Stream

A uniform flow field with no change in velocity or direction can be described with the stream and cross stream functions, taken here as an example with selected flow velocities, as follows:

$$\psi = by - ax \qquad\qquad \phi = bx + ay$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = b = .5\ meters/sec$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = a = .1875\ meters/sec$$



In this flow field, the horizontal component of the flow, **u** = b, is taken as equal to .5 meters per second to the right and the vertical component, **v** = a, is taken as equal to .1875 meters per second upwards. The total flow velocity is equal to (b^2 + a^2)^.5 = .534 meters per second. The angle of inclination of the stream lines is equal to the arctangent of a/b, in this case, 20.56 degrees.The area of the flow field is 8 meters by 8 meters. The stream lines and cross stream lines are straight and do not converge or diverge. The dimensions of the ψ and ϕ functions are meters squared per second and the loci of their zero value lines cross each other at the origin. The increment of increase of each to the right and upwards is taken as .5 m^2/sec per lateral distance of 1/ cos 20.56° = 1.068 meter which is equal to distances on the horizontal and vertical scales of 1 meter.

The functions satisfy the continuity and irrotationality conditions identically since the space derivatives of the uniform flow are zero:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 - 0 = 0$$