Two Dimensional Wing and Blade Mathematical Theory

Detailing and Extending Material in Standard References

Part 2

The Fluid Dynamics Velocity Potential "Phi" = ϕ (x, y)

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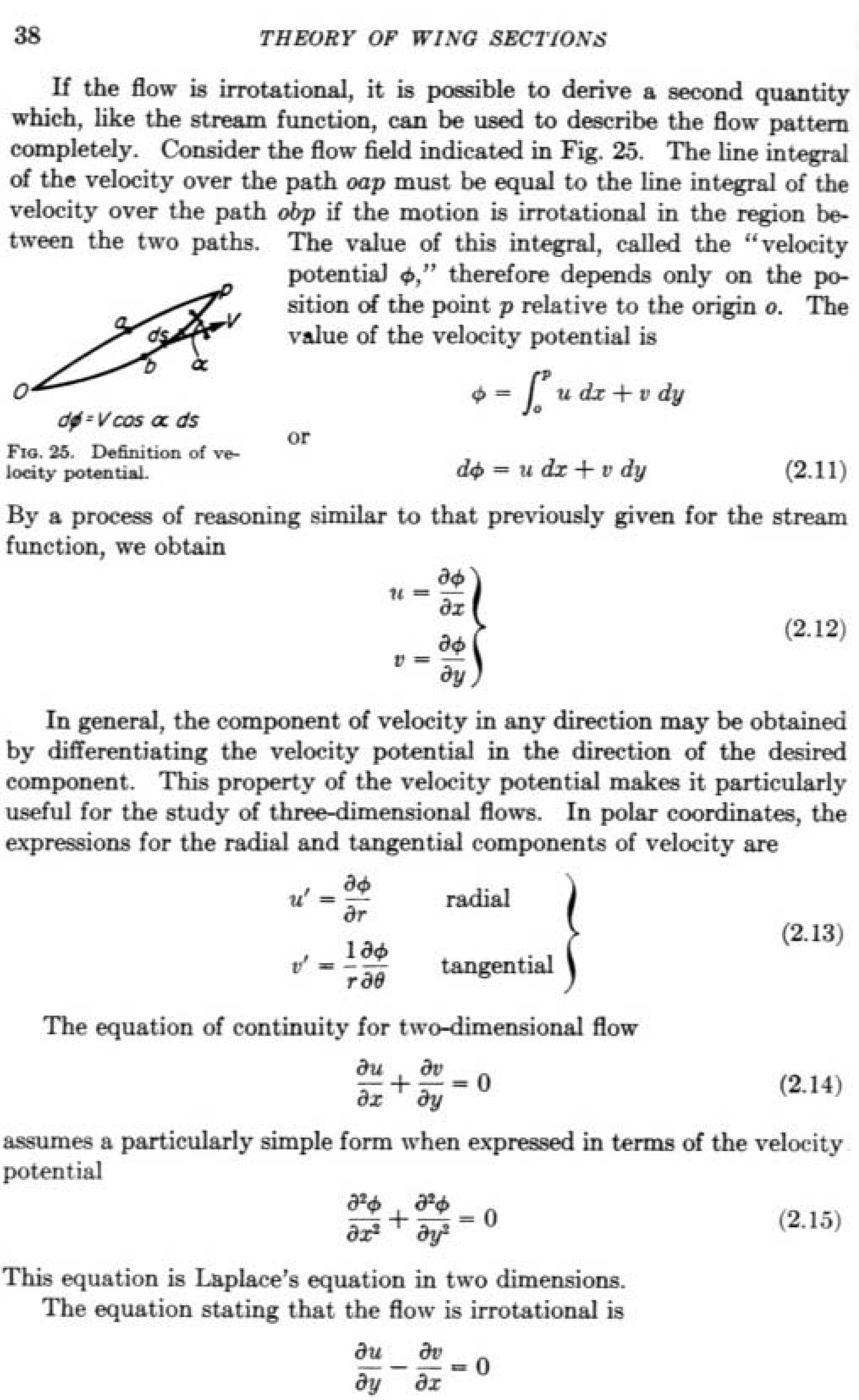
Continuing with the description of the flow field in two dimensional coordinate systems, the stream function, ψ, was covered in Part 1 and its supplementary function, the velocity potential, ϕ, will be covered here. As mentioned earlier, constructing these functions using ordinary algebraic expressions at a basic level, the flows described are elementary but, by the principle of superposition, one added on top of another, these functions can be made to describe innumerable flow geometries.

The Velocity Potential or the "Cross Stream Function"

The velocity potential is a function whose arguments, for fixed values, follow paths at right angles to the streamlines in the flow as opposed to the stream function whose arguments, for fixed values, follow paths parallel to the streamlines in the flow. Copied in below as Figure 1 is a page that defines the velocity potential from the text referenced for this subject, that is, Abbott and von Doenhoff, *Theory of Wing Sections*, Dover Publications, New York, 1959, LOC # 60-1601, Chapter 2, "Simple Two Dimensional Flows", Article 2.5, "Description of Flow Patterns", page 38. On the page preceding, the discussion covers the important assumption that is required for this definition, that of irrotationality of the flow. The angular velocity of all flow elements in any fluid can be changed only by the application of tangential, i.e. shearing, forces, to them, which are minimal in fluids with low viscosity. The concept of "vorticity" is also introduced, which is equal to twice the angular velocities at any location within the flow, and this leads to the definition of the circulation, which is the vorticity times the area of the flow elements under angular rotation. It is important to remember that irrotational flows may maintain their irrotationality even when turning corners or doubling back upon themselves, provided all flow elements slip past one another with no tangential forces exerted upon one another.

The velocity potential is so named for reasons that are not clear, since both it and the stream function can describe the flow stream satisfactorily and, by means of their derivatives, the flow velocities in it as well. Pressure gradient fields may coincide with velocity potentials although with different dimensions. Better names for both ψ and ϕ would be the "stream and velocity function" and the "cross stream and velocity function", respectively, and it might suffice to just identify the velocity potential, ϕ, as the "cross stream function", which is occasionally done in this treatment. The key to understanding this function is given in its definition at the top of the page in Figure 1. *The velocity potential or the cross stream function, ϕ, is equal to the average of the component of the flow moving parallel to and along a path from the origin of the co-ordinate system to the point whose co-ordinates, (x,y), are its arguments multiplied by the length of this path.*



If the flow is irrotational, it is possible to derive a second quantity which, like the stream function, can be used to describe the flow pattern completely. Consider the flow field indicated in Fig. 25. The line integral of the velocity over the path *oap* must be equal to the line integral of the velocity over the path *obp* if the motion is irrotational in the region between the two paths. The value of this integral, called the "velocity potential ϕ," therefore depends only on the position of the point p relative to the origin o. The value of the velocity potential is

$$\phi = \int_o^p u\,dx + v\,dy$$

or

$$d\phi = u\,dx + v\,dy \tag{2.11}$$

FIG. 25. Definition of velocity potential.

By a process of reasoning similar to that previously given for the stream function, we obtain

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial x} \\ v &= \frac{\partial \phi}{\partial y} \end{aligned}\right\} \tag{2.12}$$

In general, the component of velocity in any direction may be obtained by differentiating the velocity potential in the direction of the desired component. This property of the velocity potential makes it particularly useful for the study of three-dimensional flows. In polar coordinates, the expressions for the radial and tangential components of velocity are

$$\left.\begin{aligned} u' &= \frac{\partial \phi}{\partial r} && \text{radial} \\ v' &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} && \text{tangential} \end{aligned}\right\} \tag{2.13}$$

The equation of continuity for two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.14}$$

assumes a particularly simple form when expressed in terms of the velocity potential

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.15}$$

This equation is Laplace's equation in two dimensions.

The equation stating that the flow is irrotational is

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

Figure 1. The definition of the velocity potential, ϕ, in Abbott and von Doenhoff

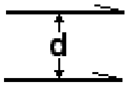
A Fixed Value of ϕ in the Flow Field is a Line at Right Angles to the Flow Stream

Here is what this implies. If two points have the same value of ϕ, then the average values of the components of flow parallel to the path of a path between the origin and each of them times the length of each path must be the same. If this is so, then the average value of the component of flow parallel to the path of any path *between them* must be zero, which is, of course, equal to the difference between the two values of ϕ. If this is so, then a path must exist between them on which the flow velocity does not have nonzero (+ or -) components all along its length that net out and thus is actually itself zero ($\cos\theta = 0$) and at right angles to the path all along its length. A proof using math is provided below.

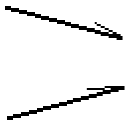
The Assumptions of Incompressibility and Irrotationality for ψ and ϕ

Questions arise in practical applications making use of these functions about whether the assumptions of incompressibility and irrotationality required for them can properly be made. For the case of aviation and wind energy involving air as the flow medium it has been determined that for subsonic flows and where flow deflection angles are not great, corrections are normally not necessary and computations using them have been found to be accurate, although questions such as this never stay still for long. In their favor, it must be said, what is obtained from their use in analyses is the ability to determine the flow streams and their velocities up to large distances removed from the airfoil surfaces and thus to predict well the flow velocity profiles over such distances and the velocity profile averages, something of great benefit. For it is the motion of the large body of air that is passing by both near and far from airfoil surfaces that is responsible for the forces that are seen by them.

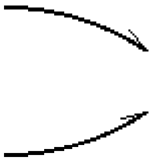
As a brief addendum here to Part 1 on the stream function, it is to be noted that carrying the derivations of the stream function beyond the first derivative provides insights into the accelerations of the flow and even the rate of change of the accelerations as may be seen in the following drawings of stream lines:



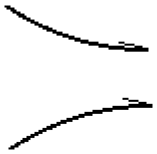
The distance, d, between streamlines is inversely proportional to the flow velocity. This corresponds to the rate of change or the derivative of the stream function at right angles to the flow.



Tapered streamlines reducing to a smaller distance of separation represent a flow acceleration. This corresponds to a fixed value of the second derivative of the stream function at right angles to the flow.



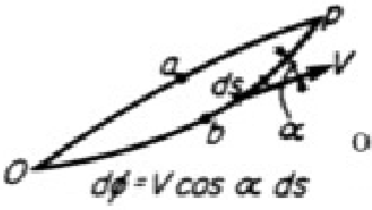
Streamlines with a curvature while narrowing represent a flow acceleration with the acceleration itself increasing. This corresponds to a third derivative of the stream function at right angles to the flow.



Streamlines with a curvature while narrowing that is reducing the rate of narrowing represent a flow acceleration with the acceleration itself decreasing. This corresponds to a negative value of the third derivative of the stream function at right angles to the flow.

Invariance of ϕ Over All Line Integrals Between Points Assuming Irrotational Flow

The definition of the velocity potential as excerpted as Equation 2.11 from Figure 1 above is:

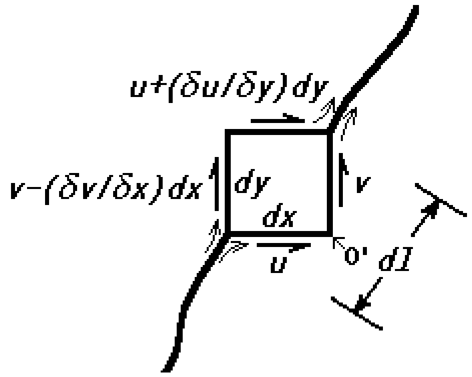


or

$$\phi = \int_o^p (u\,dx + v\,dy)$$
$$d\phi = u\,dx + v\,dy \qquad (2.11)$$

As was mentioned in Part 1 for a similar definition for the stream function by means of a line integral, care must be taken in integrating using two differentials, dx and dy, under the one integral sign. The integrand, u dx + v dy, is noted as being the scalar dot product of the flow velocity vector times the differential path vector and according to vector algebra this is the same as the relation shown under the drawing, that is, $d\phi = V \cos \alpha\, ds$. This clearly shows the integration as being of the component of the velocity V parallel to the path all along its length and the integral is equal to the average of this velocity component from the origin to the point *p* times the length of the path.

As it stands, the requirement for this as a definition that the integral be invariant over all paths taken between the origin and the point *p* cannot yet be assumed. The relation of flow irrotationality has not been applied to it. The correct expression for this integral in a more general form would include two more terms. These terms cover the possibility of the values of the velocities changing during passage through a differential segment of the line, dl, described by the dx and dy differentials as seen in the following drawing:



$$\varphi = \int((u+(\delta u/\delta y)dy)dx + (v-(\delta v/\delta x)dx)dy)$$
$$= \int((udx + vdy) + \underbrace{(\delta u/\delta y - \delta v/\delta x)}_{=\,0}dx\,dy)$$

Since the flow is assumed from the lower left to the upper right in following the path line differential segment, an origin is taken at the corner labeled o' in the drawing. The flow moving along the path going around to the right, which is on the horizontal and vertical axes of the segment, has no differential velocity elements in it and so may be integrated as in Eq. 2.11. The flow moving along the path going around to the left is equal to the original v component minus its differential decrease to the left of the vertical axis plus the original u component plus its differential increase above the horizontal axis. The integral may then be written including these two differential velocity terms and the integrand terms grouped as shown.

Now if the following relation under the assumption of irrotational flow,

$$(\partial u/\partial y) - (\partial v/\partial x) = 0$$

holds for all paths in the flow field between the two points, it may be applied. The third and fourth differential terms together in the integrand cancel each other out as indicated above and the Eq. 2.11 on the reference page is then found to be the result again. The integral integrands are then invariant in all these line differentials and so the integral is invariant overall. Thus the velocity potential for each point in the flow field has a single value.

Math Proof of Velocity Potential as Creating Cross Stream Lines

Now that it is ascertained that the velocity potential has a single value for each point in the flow field under the given assumptions, the proof can be given for the above statement that the locus of points generated by all the velocity potentials which have a fixed value is a cross stream line, that is, a line at right angles to the flow at all points along the line.

First, the differential equation for the stream path of a flow element in Cartesian co-ordinates with velocity u in the x direction and v in the y direction is:

$$\frac{dy(x)}{dx} = \frac{v(x,y)}{u(x,y)}$$

that is, the slope of the function y(x) is equal to the vertical component of the velocity divided by the horizontal component of the velocity at each point. Now for cross stream paths in Cartesian co-ordinates, the slopes at every point must be at right angles to the slopes given in the above equation. Such slopes are the algebraic negative reciprocals[1]. Thus the differential equation would be altered as follows:

$$\frac{dy(x)}{dx} = \frac{-u(x,y)}{v(x,y)}$$

As mentioned in Part 1 for the stream function, each y(x) function describes only one flow streamline and two velocity functions, u and v, are needed to describe the velocity of flow for the

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[1] A good description of the process of obtaining formulae whose plots are orthogonal (at right angles) to the plots of given formulae is available in Reddick, H.W. and Miller, F.H., *Advanced Mathematics for Engineers*, Third Edition, 1960, John Wiley & Sons, Inc., New York and London, LOC #55-6102, Chapter 1, Article 7(a), page 19 et seq.

two co-ordinates of the streamline. The velocity potential replaces this with one double argument function that provides all the cross stream lines in the flow field and also whose derivatives with respect to the co-ordinate axes are equal to the velocities within the streamlines crossed as seen in Eqs. 2.12 in the text of Figure 1 above.

The expression may be rearranged as:

$$\frac{dx}{v(x,y)} = \frac{-dy}{u(x,y)}$$

This states that for lines at right angles to the flow streamlines the displacement along each of the two co-ordinates is proportional to the flow velocities along each of the opposite co-ordinate axes.

The derivatives of the velocity potential with respect to the x and y co-ordinates are substituted into this equation for u and v and a few algebraic operations performed. What results is an expression which is equal to zero and is identical to the total differential of ϕ:

$$v = \frac{\partial \phi}{\partial y} \qquad u = \frac{\partial \phi}{\partial x}$$

$$\frac{dx}{\partial \phi / \partial y} = \frac{-dy}{\partial \phi / \partial x}$$

$$(\partial \phi / \partial x) dx = -(\partial \phi / \partial y) dy$$

$$(\partial \phi / \partial x) dx + (\partial \phi / \partial y) dy = 0 = d\phi$$

Hence the value of ϕ is invariant for lines at right angles across the stream line represented by the velocities u and v.

Distance Between Cross Stream Lines Related to Velocities

In drawing cross stream lines described by velocity potentials with equal value differences between them, the variation of the velocities of the flow can be determined by the distances between the cross stream lines, wide distances for slower flow and narrow distances for faster flow. This would be expected for stream lines but it also is true for the cross stream lines obtained from the velocity potentials. It is in fact possible to calculate the velocities within the streamlines by means of measuring the distances between the cross stream lines at the points of intersection:

$$V = \sqrt{u^2 + v^2} = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} \simeq \Delta \phi(x,y)/d$$

where:

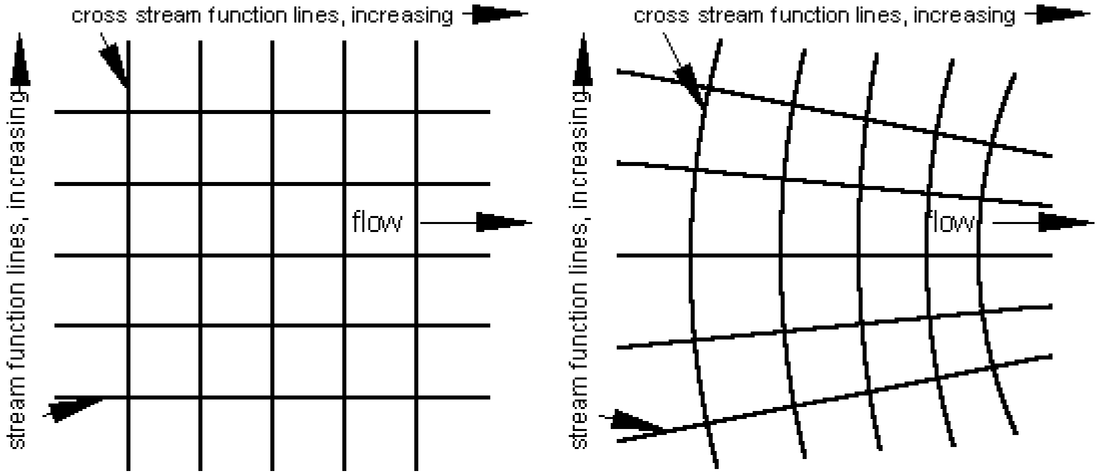
V = total flow velocity

d = distance between cross flow lines of velocity potential

It should be re-emphasized that these are cross flow lines and not streamlines but even so the distance between them is a measure of the flow velocities within the streamlines.

The Stream Function and Velocity Potential Lines of the Flow Stream

The stream and velocity potential cross stream lines for two cases of flow, one steady and the second with a fixed acceleration, are depicted in the drawings below:



As the velocity increases, the distances between both the stream lines and the cross stream lines reduces. Other considerations, such as second and third derivatives of the stream functions have their counterpart in the behavior of the velocity potential functions as well.

The dimensions of both stream functions and velocity potentials are both length squared per unit time, for example, meters squared per second. It can often be assumed that a mass density factor, ρ, is hidden within the definition in three dimensions, in which case, the dimensions would be mass units per unit time, for example, kilograms per second.

These details are covered in order to gain a complete understanding of the velocity potential, for which a suitable alternative name might be the cross stream function in its role as paired with the stream function. Lines of equal stream function values in the flow field are all parallel to the flow and lines of equal cross stream function values in the flow field cross flow lines at exactly right angles.