

# Two Dimensional Wing and Blade Mathematical Theory

## Detailing and Extending Material in Standard References

### Part 1

## The Fluid Dynamics Stream Function " $\Psi$ " = $\psi$ (x, y)

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One of the first concepts needing thorough familiarization in the field of fluid dynamics study is the description of the flow field in two dimensional coordinate systems by means of the stream function,  $\psi$ , along with its supplementary function, the velocity potential,  $\phi$ . Constructing these functions using ordinary algebraic expressions at a basic level, the flows described are elementary but, by the principle of superposition, one added on top of another, these functions can be made to describe innumerable flow geometries.

### The Stream Function

To draw a curving line on a graph that is to represent the motion of a particle of fluid from one point to another and describe it with a math function takes a formulation such as  $y = f(x)$ . This provides the location in space of only one streamline and nothing is said about the flow velocities along this streamline. For flows of an incompressible fluid in steady motion it is possible to define a single function that can provide all the streamlines anywhere in the flow field and, in addition, by its derivatives with respect to the co-ordinate space variables, the velocities of the flow within them at any point as well.

This is the stream function. Copied in below as Figure 1 is a page from the text referenced for purposes of this Part 1 subject, that is, Abbott and von Doenhoff, *Theory of Wing Sections*, Dover Publications, New York, 1959, LOC # 60-1601, Chapter 2, "Simple Two Dimensional Flows", Article 2.5, "Description of Flow Patterns", page 36. The text in Chapter 2 begins with a description of the equations of motion at a small, differential location in space in three dimensions, a treatment that can be followed readily with some background in calculus. The total derivative of the particle of fluid in motion in three dimensions requires three terms in each of three equations for a total of nine terms and, once this has been given some attention, the expressions for the motion using Newton's Law follow readily.

The key to understanding the stream function, a function providing one numeric value for the two co-ordinate variables, x and y, entered, is given at the top of the page. *The stream function,  $\psi$ , is equal to the fluid flow rate passing between the origin of the co-ordinate system and a point in the flow space whose co-ordinates, (x,y), are its arguments.*

### A Fixed Value of $\psi$ in the Flow Field is a Stream Flow Line

Here is what this implies. If two points in the flow space provide the same value for the stream function, then the flow between each of them and the origin must be the same. If the flow between each of them and the origin is the same then it must be the same flow with no net flow added or

Consider the flow across the arbitrary line  $oba$  that connects the origin  $o$  with a point  $a$  (Fig. 22). The amount of flow across this line will be the same as that across any other line connecting the two points, for example, line  $oca$ , provided that the equation of continuity holds for all points within the region enclosed by the two lines. With a fixed origin  $o$  the amount of fluid crossing any line joining the points  $o$  and  $a$  is therefore a function only of the position of the point  $a$  for a given flow pattern. The amount of fluid passing between the points  $o$  and  $a$  is given by the following expression

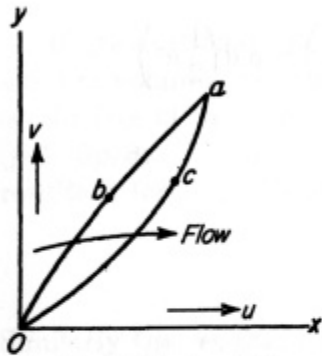


FIG. 22. Derivation of stream function.

$$\psi = \int_o^a u \, dy - v \, dx$$

or

$$d\psi = u \, dy - v \, dx \quad (2.6)$$

Because the value of  $\psi$  is independent of the path of integration, Eq. (2.6) remains valid even though  $dx$  and  $dy$  are varied independently. The general expression for the total differential of  $\psi$  is

$$d\psi = \frac{\partial \psi}{\partial x} \, dx + \frac{\partial \psi}{\partial y} \, dy \quad (2.7)$$

This equation is also valid for independent variations of  $dx$  and  $dy$ . The coefficients of  $dx$  and  $dy$  in Eqs. (2.6) and (2.7) must therefore be equal, or

$$\left. \begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad (2.8)$$

A single function  $\psi$  has therefore been found by means of which it is possible to define both components of velocity at all points in the field of flow. This function  $\psi$  is called the "stream function." Lines in the flow along which  $\psi$  is constant are called "streamlines," and these lines are the paths of motion of fluid elements for steady motion.

In general, the component of velocity in any direction may be obtained by differentiating the stream function in a direction  $90^\circ$  counterclockwise to the component desired. In polar coordinates, therefore, the expressions for the radial and tangential components of velocity are

$$\left. \begin{aligned} u' &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} && \text{radial} \\ v' &= -\frac{\partial \psi}{\partial r} && \text{tangential} \end{aligned} \right\} \quad (2.9)$$

Figure 1. The definition of the stream function,  $\psi$ , in Abbott and von Doenhoff

taken away between these two points. If no net flow is added or taken away between these two points then it must mean that all flows crossing one way between these two points are netted out by all flows crossing the other way between them and so a line can be found between them across which no flows are occurring. If a line can be found between them across which no flows are occurring then this clearly must be a line that is on a single flow line and following the flow path. Hence the conclusion: each streamline can be represented by a certain fixed value of the stream function. A proof using math is provided below.

### Invariance of $\psi$ Over All Line Integrals Between Points for Incompressible Flow

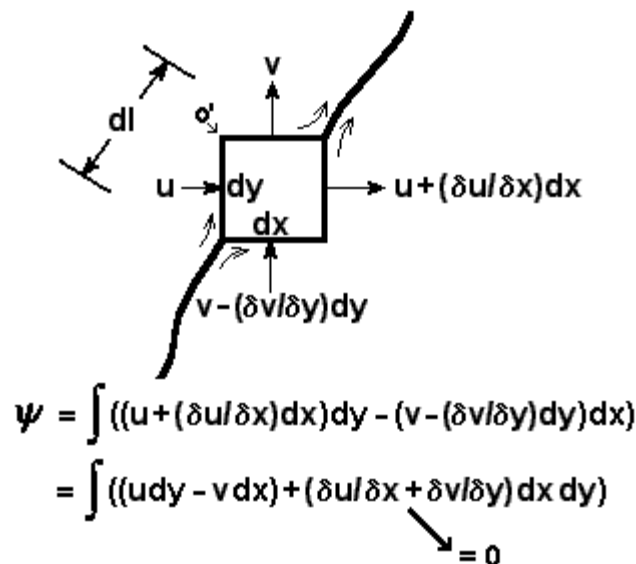
Something must be said about the definition of the stream function as it appears in Equation 2.6 in the text of Figure 1 above:

$$\psi = \int_0^a (u \, dy - v \, dx)$$

$$d\psi = u \, dy - v \, dx \quad (2.6)$$

Parentheses were added to the integrand of the integral since the integral is described as a line integral, which has the quality that just the one integral sign covers an integrand with two differentials within it,  $dy$  and  $dx$ . Care must be taken to not equate this to two integrals at the outset, one for each of the two terms in the integrand, nor to a double integral, that is, an area integral covering both differentials, first the one and then the other.

As a line integral the proper differential to use is  $dl$  and the  $u$  and  $v$  velocities are expressed as components of the total velocity. Further, as it stands as a definition of the stream function, the requirement that the value of the integral be invariant over all paths taken between the origin and the point  $a$  cannot yet be assumed. The equation of continuity has not been applied to it. The correct expression for this integral should first include two more terms. These terms cover the possibility of the values of the velocities changing during passage through a differential segment of the line,  $dl$ , described by the  $dx$  and  $dy$  differentials as seen in the following drawing:



Since the flow is assumed from the upper left to the lower right in crossing the line, an origin for

this differential element is taken at the corner labeled 0' in the drawing. The differential for the v velocity in the y direction is then negative below the co-ordinate line. The total flow volume crossing the line in going around to the left has no differential elements in it and so may be integrated as in Eq. 2.6. The total flow volume crossing the line at this differential element in going around to the right is equal to the u component plus its differential increase in moving through the small box minus the v component minus its differential increase in moving through the small box. The integral may then be written including these two differential velocity terms and the integrand terms grouped as shown.

Now if the equation of continuity,

$$(\partial u / \partial x) + (\partial v / \partial y) = 0$$

holds for all paths in the flow field between the two points, it may be applied. The third and fourth differential terms together in the integrand cancel each other out as indicated above and the Eq. 2.6 on the reference page is then found to be the result again. The integral integrands are then invariant in all these line differentials and so the integral is invariant overall. Thus the stream function for each point in the flow field has a single value.

### Math Proof of Streamline

Now that it is ascertained that the stream function has a single value for each point in the flow field under the given assumptions, the proof can be given for the above statement that the locus of points generated by all the stream functions which have a fixed value is a flow streamline.

First, the differential equation for the path of an element in Cartesian co-ordinates with velocity u in the x direction and v in the y direction is:

$$\frac{dy(x)}{dx} = \frac{v(x, y)}{u(x, y)}$$

that is, the slope of the function y(x) is equal to the vertical component of the velocity divided by the horizontal component of the velocity at each point. As mentioned above, each y(x) function describes only one flow streamline and two velocity functions, u and v, are needed to describe the velocity of flow for the two co-ordinates in the streamline. The stream function replaces this with one double argument function that provides all the streamlines in the flow field and also whose derivatives with respect to the co-ordinate axes are equal to the velocities within these streamlines as seen in Eqs. 2.8 in the text of Figure 1 above.

The above may be rearranged as:

$$\frac{dx}{u(x, y)} = \frac{dy}{v(x, y)}$$

This states that the displacement along each of the two co-ordinates is proportional to the velocities along each co-ordinate axis.

The derivatives of the stream function with respect to the x and y co-ordinates are substituted into this equation for u and v and a few algebraic operations performed. What results is an expression which is equal to zero and is identical to the total differential of  $\psi$ :

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{dx}{\partial \psi / \partial y} = \frac{dy}{-\partial \psi / \partial x}$$

$$-(\partial \psi / \partial x) dx = (\partial \psi / \partial y) dy$$

$$(\partial \psi / \partial x) dx + (\partial \psi / \partial y) dy = 0 = d\psi$$

Hence the value of  $\psi$  is invariant for the streamline represented by the velocities  $u$  and  $v$ .

### Distance Between Streamlines Related to Velocities

In drawing streamlines described by stream functions with equal value differences between them the variation of the velocities of the flow can be determined by the distances between the streamlines, wide distances for slower flow and narrow distances for faster flow. This would be expected in any event but the use of stream functions makes it possible to in fact calculate the velocities by means of measuring the distances between the flow lines:

$$V = \sqrt{u^2 + v^2} = \sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2} \simeq \Delta \psi(x, y) / d$$

where:

$V$  = total flow velocity

$d$  = distance between streamlines

### The Dimensions of the Stream Function

In calculating the flow velocities from the distance between flow streamlines, it should be noted that the stream function has dimensions. The dimensions of  $\psi(x, y)$  are length squared (or area) divided by time or the area swept out by the flow moving between the co-ordinate origin and the point  $(x, y)$  in a given amount of time.

These details are covered in order to gain a complete understanding of the stream function and its definition as a foundation for analysis work that requires its use.